Fractals: the Mandelbrot set

3rd year project 2009/10

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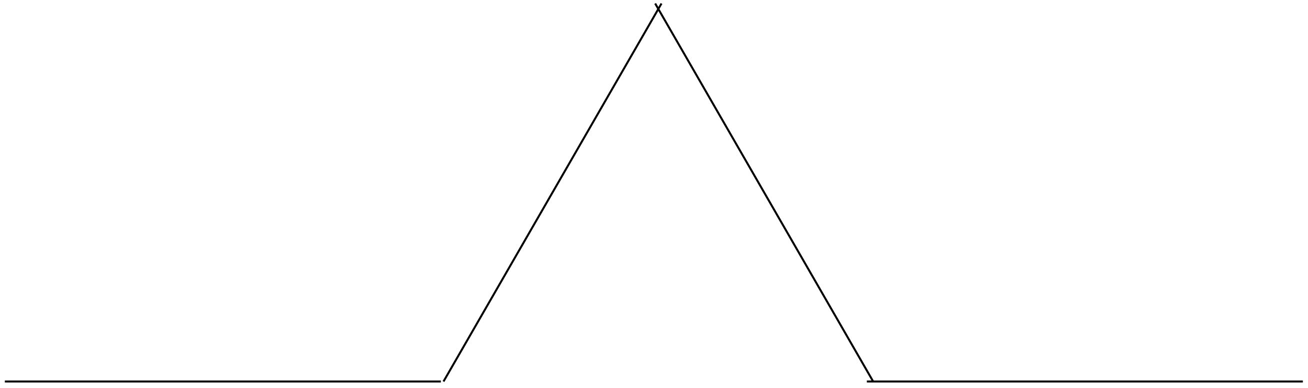
- The first objective of this project is to familiarise yourself with the main properties of some mathematical objects called fractals.
- I will give you some photocopies from chapter 11 of the book *«Non linear dynamics and chaos by Stephen H. Strogatz, Addison Wesley (1994)»* but there are many other books and also a lot of material in the internet that can be useful.
- Most books on dynamical systems or chaos will have a section on fractals (check our library)
- Once you have read a bit about fractals you will probably be able to answer the following questions:

- Fractals are typically described as being self-similar or quasi self-similar. What does this mean?
- Are there any fractals really existing in nature or are they purely mathematical objects? If you think that fractals really exist in nature, can you give some examples?
- What is the fractal dimension? Do you know any examples of fractal dimensions?
- Many fractals can be constructed by means of iterative or recursive procedures. How would one describe such procedures?

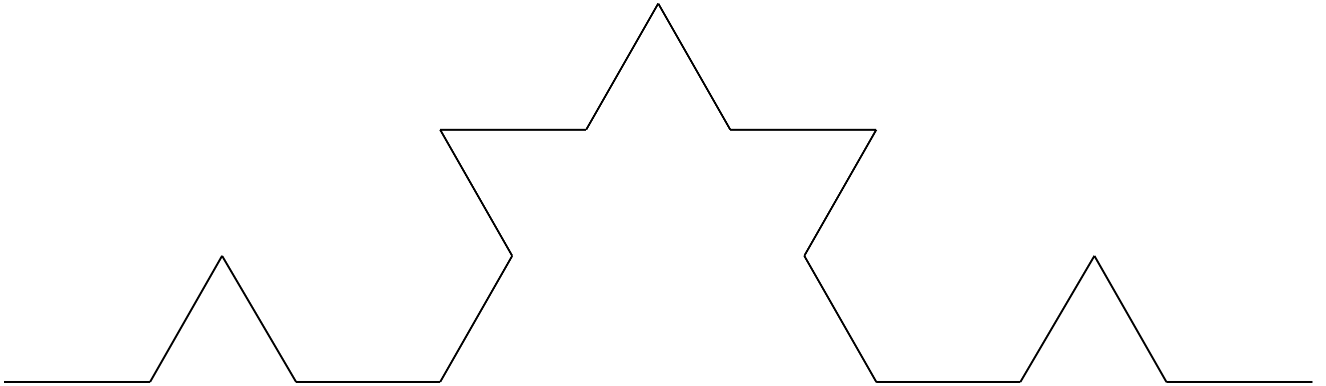
- One way of answering these questions is to look at a particular example.
- Let us look at *the Koch curve*.This is a typical example of a fractal which is constructed by means of an iterative procedure.
- Iterative procedures are usually easy to program. They will involve repeating the same kind of operation many times which in programmation has to do with loop structures (see 1st year programming!).

The Koch curve after 0 iterations

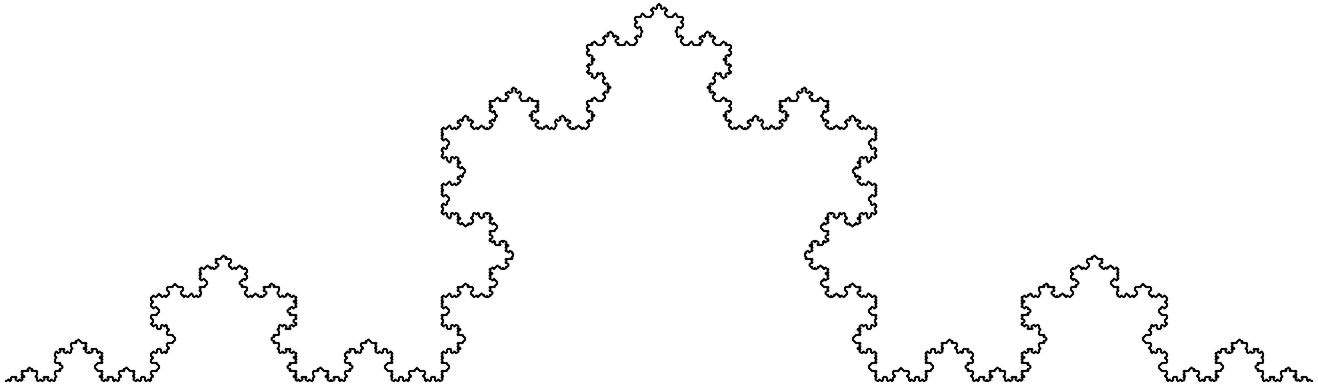
The Koch curve after 1 iteration



The Koch curve after 2 iterations



The Koch curve after many iterations



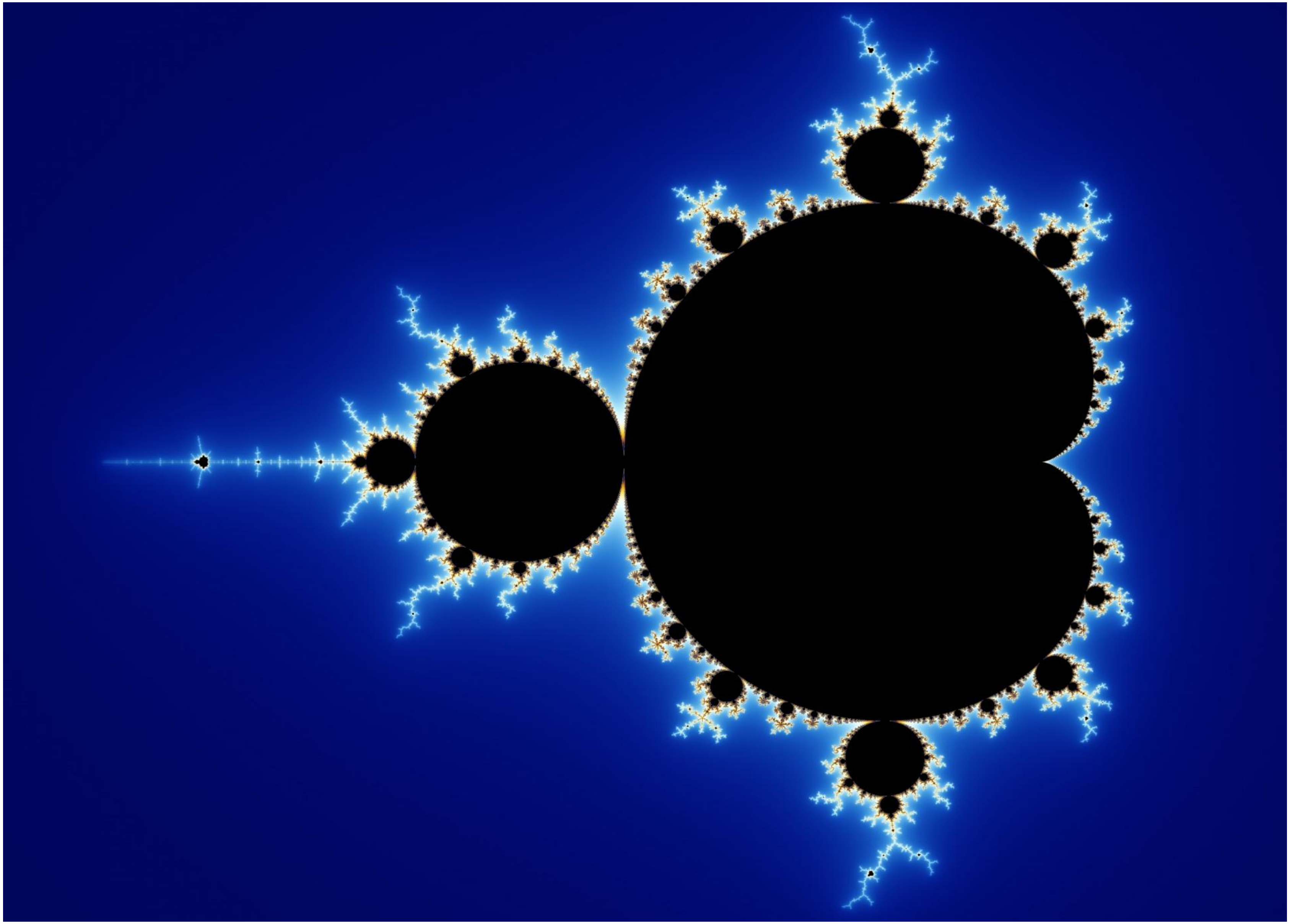
- Despite being easy to construct, the Koch curve has a lot of unusual properties, which are shared by many other fractals.
- It has a fractal dimension of $\ln 4/\ln 3=1.26$ (bigger than its topological dimension $d=1$)
- It has infinite length!
- It is continuous everywhere but differentiable nowhere!
- The fractal dimension of the Koch curve can be obtained by simple similarity arguments or also numerically by computing the box counting dimension.

- There are two types of fractals which are particularly famous: *The Mandelbrot Set and the Julia Sets.*
- They are closely related to each other and they are also constructed by means of an iterative procedure.
- In fact they are both related to the quadratic iterative map:

$$z_{n+1} = z_n^2 + c,$$

where c, z_n and z_{n+1} are complex numbers.

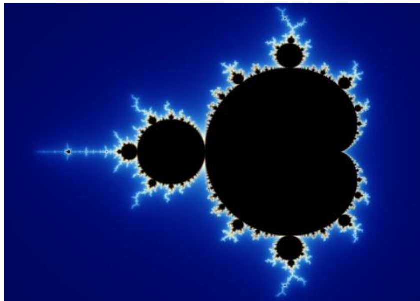
- The Mandlebrot and Julia sets are famous for their beautiful structure and it is quite astonishing that they are related to so simple an equation!



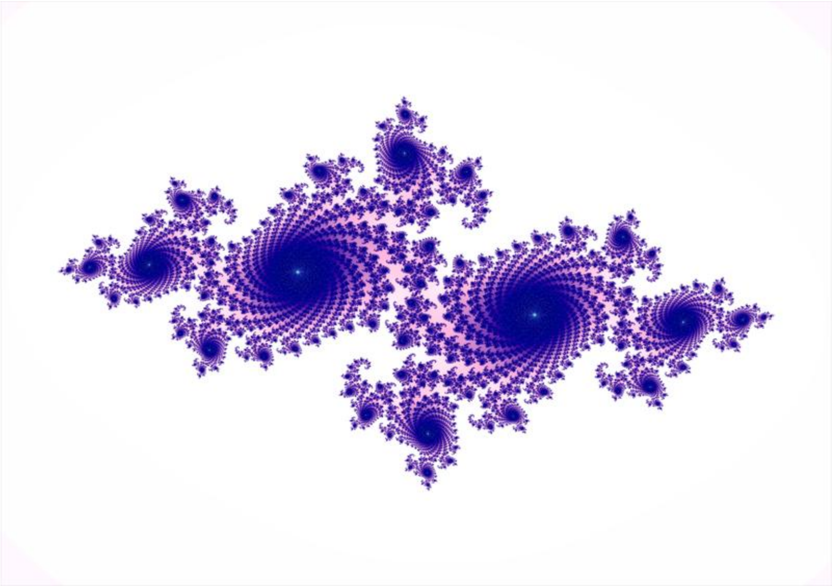
http://en.wikipedia.org/wiki/Image:Mandel_zoom_00_mandelbrot_set.jpg

- The Mandelbrot set is the black region of the picture. The set of points in that region is obtained from the equation $z_{n+1} = z_n^2 + c$, as follows:
- Let us choose an arbitrary complex number c and fix $z_0=0$. Then, iterating the equation we get:

  $z_0=0$, $z_1=c$, $z_2=c^2+c$, $z_3=(c^2+c)^2+c$….
- After iterating many times, for most values of c, $|z_n|$ will get larger and larger. But for some values of c, $|z_n| <2$.
- Whenever the latter condition is fulfilled, we will draw a dot in an x-y diagram where we represent the real part of c in the x-axis and the imaginary part of c in the y-axis.
- The set of points c in the xy-plane that satisfy this condition are the Mandelbrot set.



- What about Julia sets? They are very closely related to the Mandelbrot set. In fact there is a Julia set for every point in the Mandelbrot set!
- For a point c in the Mandelbrot set, the associated Julia set is the set of points z_0, z_1, z_2, z_3 … that are obtained by iterating the map $z_{n+1} = z_n^2 + c$, starting with $z_0 = 0$.



The Julia set for
C= -0.7268+ i 0.1889

http://en.wikipedia.org/wiki/Julia_set

Some ideas for your project:

- There are many things that one can write about fractals. It is not difficult to produce an essay-type project in this subject
- However, the aim of every project is to do something original and in a topic like this, there are many things that can be done which involve a little bit of programmation.
- I would like you to at least try some of this things, starting with a very simple problem and then adding more and more complexity.
- Even if programming is not your main strength you will always be able to ask me questions and I will provide you with examples that you can then modify/adapt.

- Let us start by writting a very simple program for the Mandelbrot set:

```
Sub mandelreal()
'Given a real number c, determines whether or not it is in the set
r1 = "Enter c:"
m = "Is c in the Mandelbrot set?"
a = InputBox(r1, m)
z = 0
k = 0
For k = 1 To 50
zit = z ^ 2 + c
z = zit
If zit > 100000 Then
Exit For
End If
Next k
If z < 2 Then
Range("A1").Value = "the point" & c & "is in the set"
Else
Range("A1").Value = "the point " & c & " is not in the set"
End If
End Sub
```

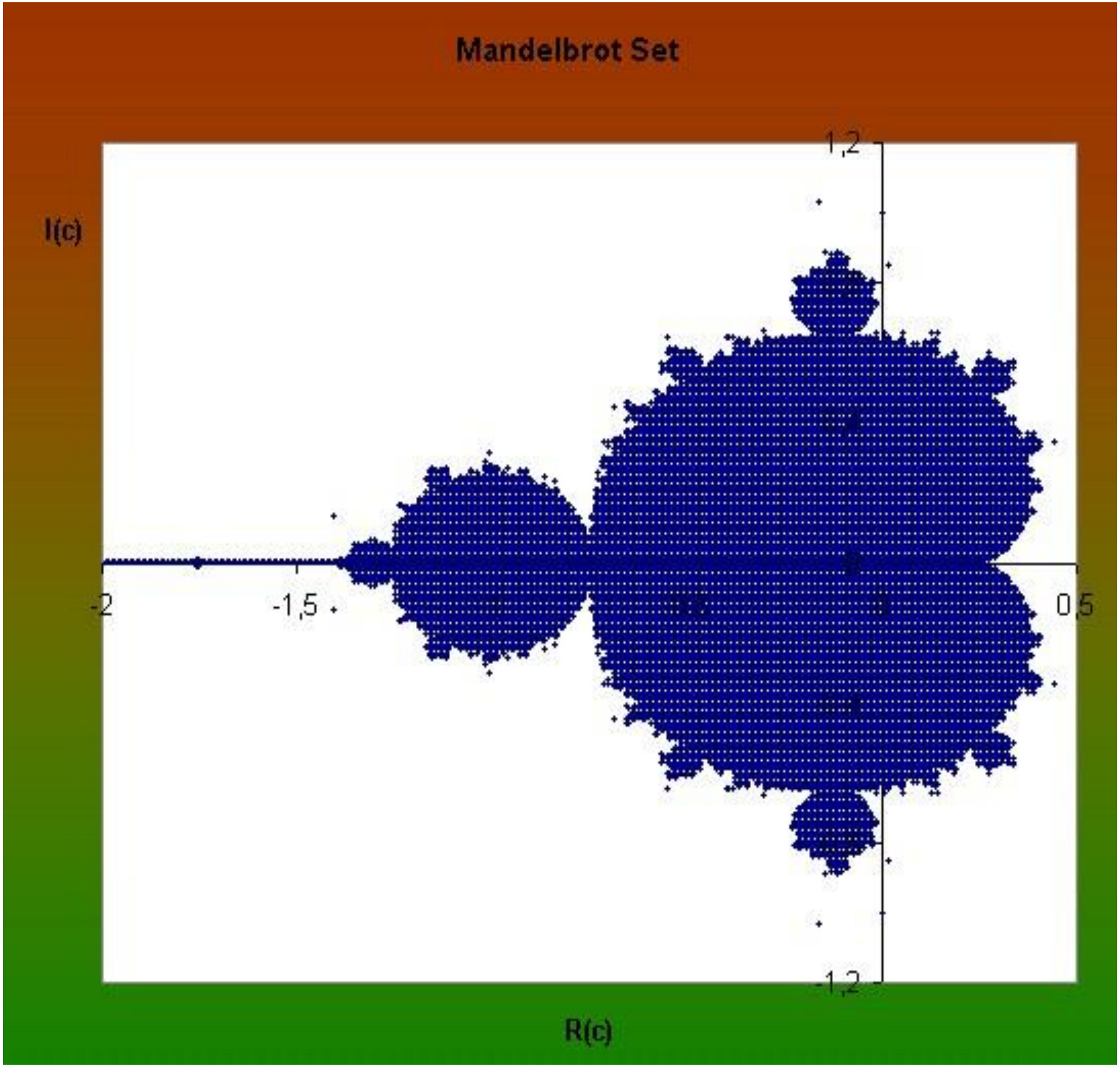
Value of z_0

Iteration

If z_n gets very large, stop iteration

If $z_n < 2$, c is
In the set

- In this program we are looking at points c and z_n which are real
- Problem: generalize this to the case when c and z_n are complex
- Write a program that does the same but checks many values of c
- Write a program that does the same for many values of c and whenever c is in the Mandelbrot set, it writes its real and imaginary part in some cells of the Excel worksheet.
- This would generate a set of points that you can then plot. If you check a large number of points, you will get something like this…


Mandelbrot Set
I(c)
1,2
-1,2
-2
-1,5
0,5
R(c)

- Once you have written such a program, you can modify it to look at just some small part of the Mandelbrot set, so that you see the structure in more detail
- It would also be easy to write a similar thing for a Julia set
- Before you start the project, you may want to learn some LaTeX. See

http://www.staff.city.ac.uk/o.castro-alvaredo/myprojects.htm