The butterfly effect: what is chaos?

3rd year project 2007/08

Supervisor: Olalla Castro Alvaredo

- The aim of this project is that you learn about the mathematical concept of chaos and study it in the context of some particular examples.
- In mathematics chaos describes the behaviour of the solutions of certain non-linear equations. The main characteristic of these solutions is their extreme sensitivity to the initial conditions.
- The kind of equations we are talking about here are basically of two types: discrete or continuous evolution equations.

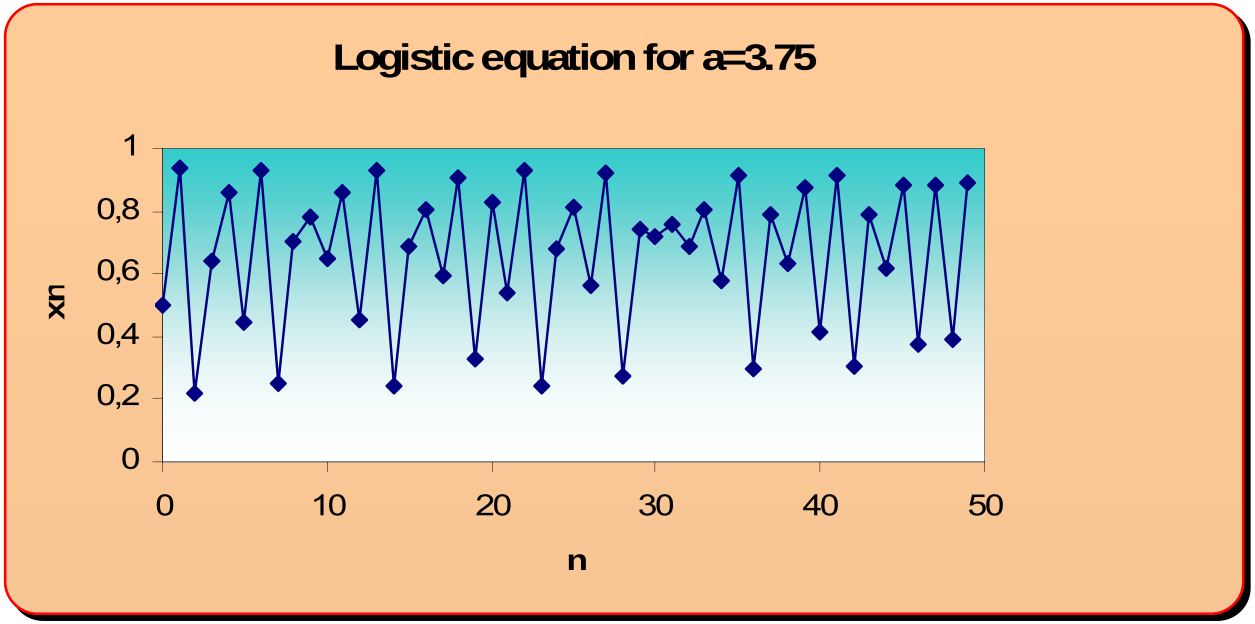
- Discrete means that the equations depend on a variable that only takes discrete values like 0,1,2,3....
- The most famous example of this kind is the logistic equation. It looks like this:

$$X_{n+1}=a\,X_n(1-X_n) \text{ with } n=0,1,2,3\ldots$$

- You can imagine that X_n describes the population of animals of a certain kind in the year n. In year 0 the population will be a certain value X_0 and by iteratively using the equation we can find the populations in successive years. The value X_0 is the initial condition.

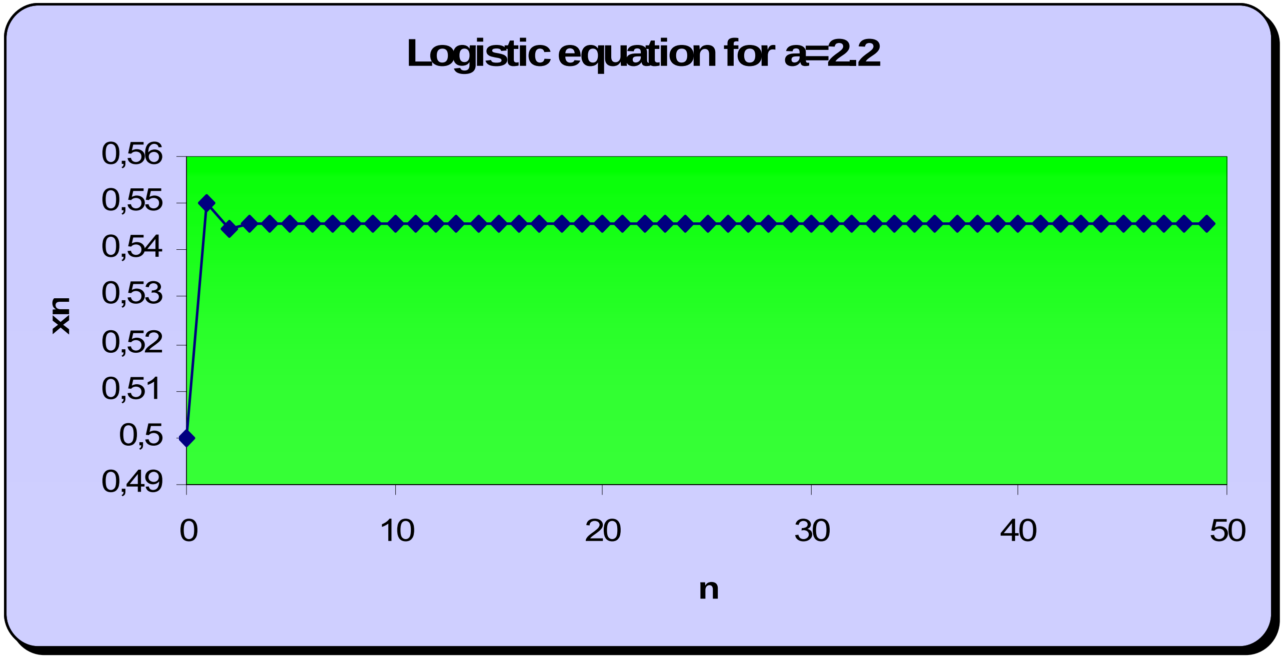
- It turns out that small changes on the initial value X_0 can make the value X_n when $n \to \infty$ change dramatically. This means that the evolution of this population in the long-term will be dramatically different whether we start with, let's say 11 individuals than with 12. That seems counter-intuitive but is precisely what is found.
- In fact for particular values of the parameter a the systems is chaotic which means that the population does not converge to a particular number after many years but varies chaotically from one value to the next.

- It is quite surprising that such a simple equation can desplay such complex behaviour. In addition, because the equation is so simple we can use easy techniques to find the fixed points of the system, study their stability and predict the precise value of a for which chaos starts. If you have done dynamical systems, you should be able to study this quite easily.
- There many different types of diagrams that you can produce to illustrate the behaviour of the solutions of this equation.
- For example you can plot X_n against n.



Chaotic region: the system does not converge to any fixed point.

For 1<a<3 the system quickly evolves towards a fixed point.



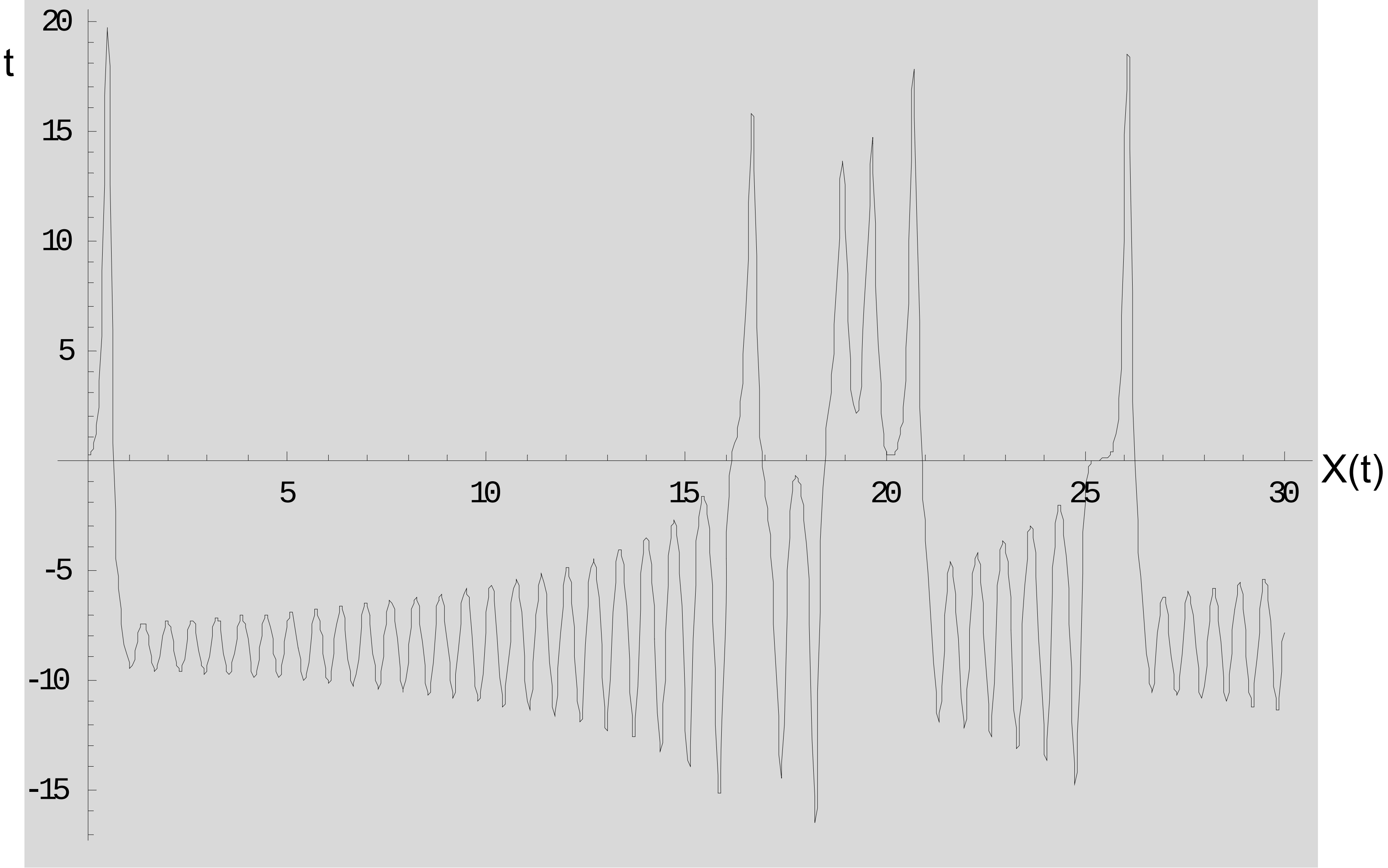
- The analysis of the solutions of the Logistic equation is a very nice project. Many different things can be done and in the book «Nolinear Dynamics and Chaos by Steven H. Strogatz », at the end of chapter 10, there are many exercises proposed, which you could try to solve for your project. This chapter also contains a detailed analysis of the Logistic equation.
- Some nice plots can be produced (like the ones I showed you, or more complicated things) and it will be nice if you can incorporate some numerical analysis to your project. You can use Excel and/or VBA for this.
- You could also analyse other equations (some are proposed in the book).

- Let us now turn to continuous evolution equations. In this case we have a continuous rather than a discrete variable involved. In many cases this is a variable t which we can interprete as time. You also have studied nonlinear equations of this type in Dynamical Systems and should know how to analyse stable fixed points and so on. But the solutions to some of these equations also exhibit chaotic behaviour.
- A very famous example of this are the Lorenz equations.

$dx/dt = a\,(y-x)$, $dy/dt = r\,x - y - x\,z$, $dz/dt = xy - bz$

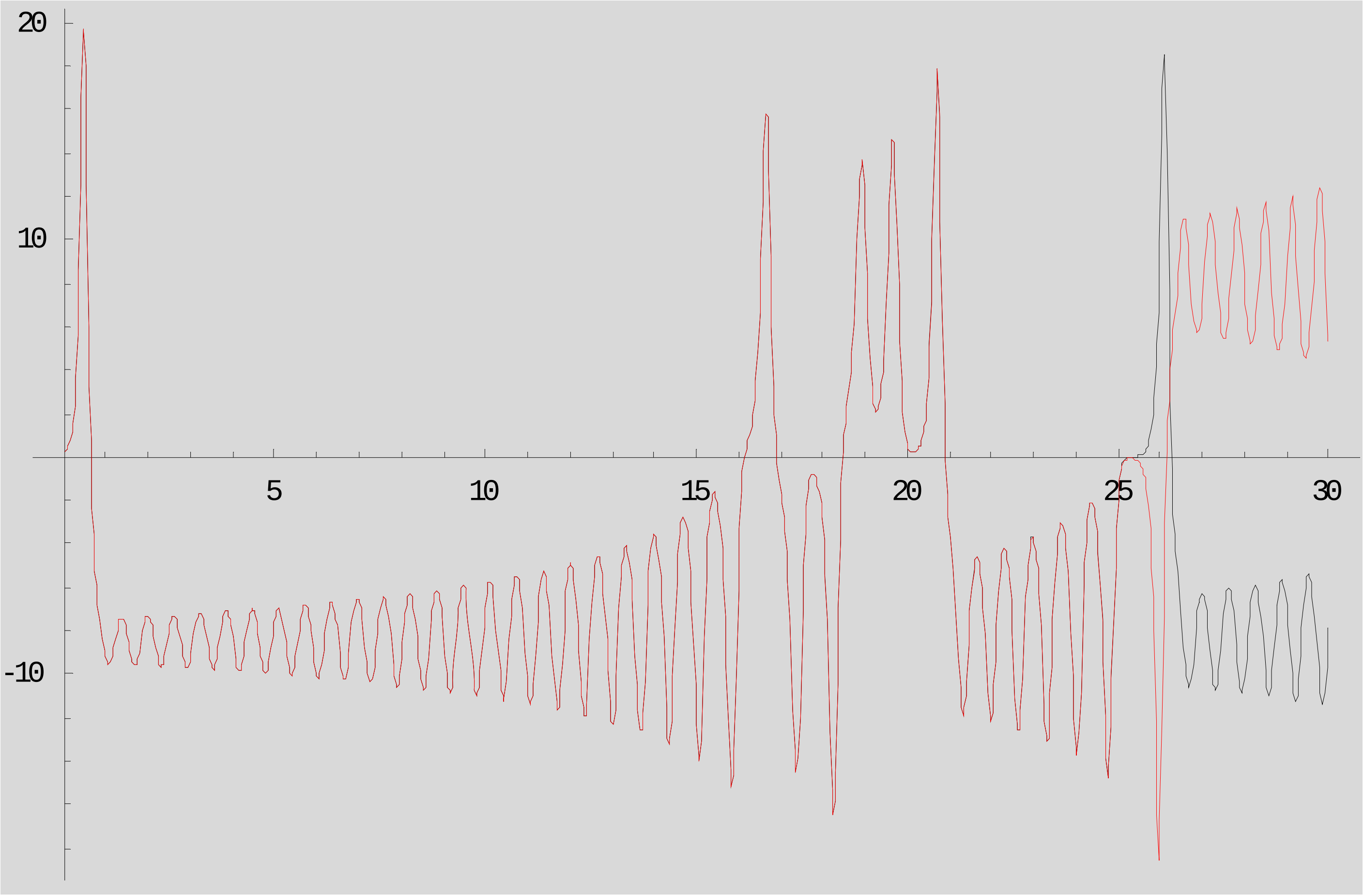
- Here x,y,z are functions of t and a,b,r are constants.
- These equations were first employed as a simplified version of the equations describing the weather conditions. The functions x,y and z are related to the atmospheric temperature and pressure!

- Lorenz was trying to solve these equations on a computer when he found that a very small change on the values $x(0)$,$y(0)$, $z(0)$ (the intial conditions) lead to an enormous change of the values of $x(t)$, $y(t)$ and $z(t)$ for some time t. Intially he thought this was some kind of mistake, but after running the program many times he saw that this strange behaviour was happening again.
- You can see that yourself by repeating Lorenz computation: You fix some initial values, e.g. $x(0)=y(0)=z(0)=0.2$ and solve numerically for many values of t. You get something like:



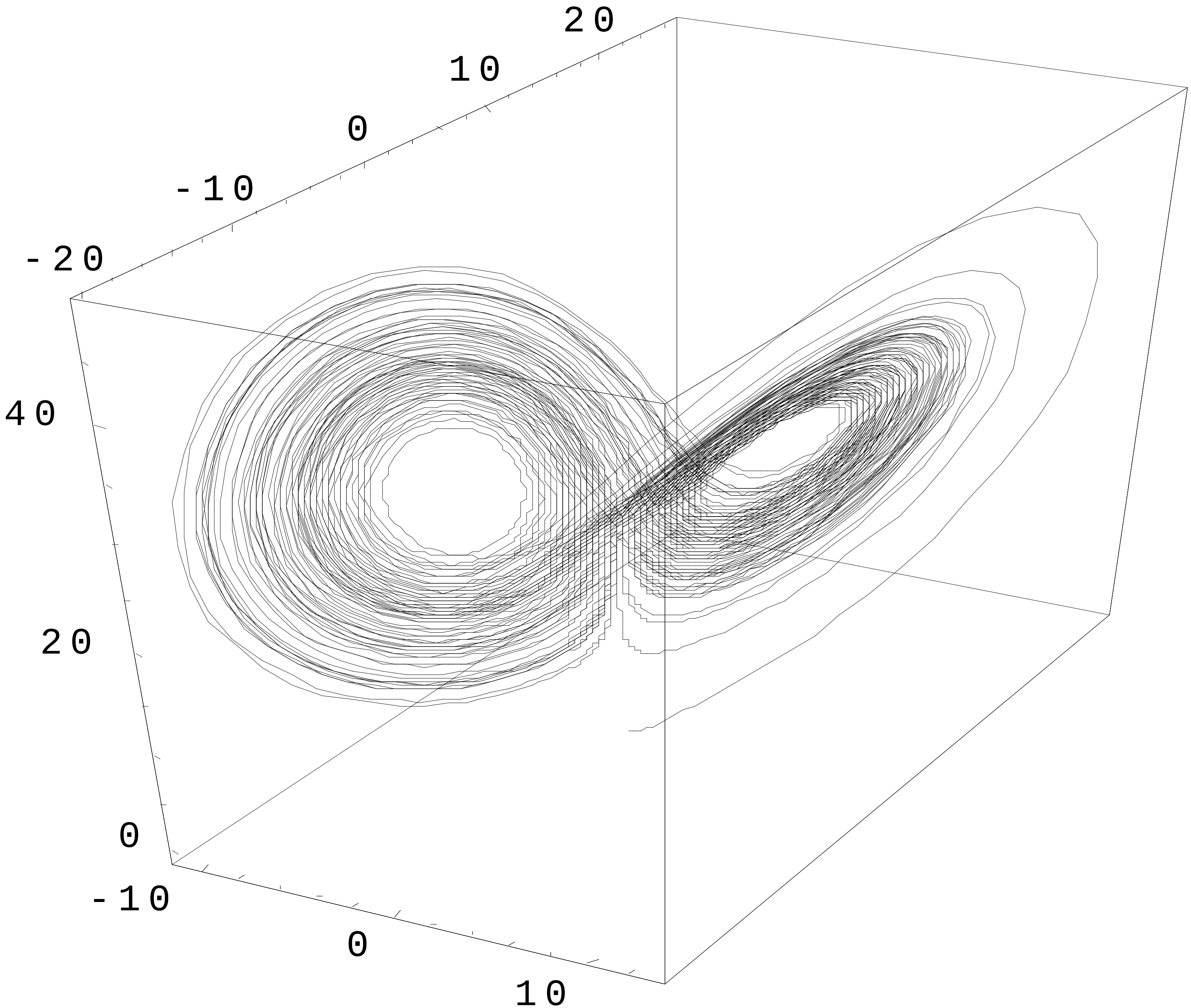
Now we change the initial conditions slightly $x(0)=y(0)=z(0)=0.200001$, and solve again…

- If we plot both solutions in the same graph you can see that after some time, the two start to be rather different…



- The reason is that for some values of the constants r,a and b the system of equations exhibits chaotic behaviour (that is the main reason why we can not predict the weather in the long term!)
- Sometimes people like to speak of chaos as the «butterfly effect». This comes actually from the Lorenz equations. Due to the extreme sensitivity of the equations to small changes in the initial conditions we can say that if a butterfly flaps its wings somewhere in the USA it would produce a tiny change in the atmospheric presure and temperature. This small change could produce a huge change in the evolution of the weather conditions and maybe even cause a tornado in London!

- Another connection with butterflies comes from plotting the solutions (x(t),y(t),z(t)) for many values of t in a three dimensional plot (for each value of t we get one point). If we do such a plot in the chaotic region we will get a rather strange figure which is know an «strange attractor» and has a very characteristic shape, similar to the wings of a butterfly:



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- Chapter 11 of the book mentioned before deals with the Lorenz equations and proposes to you several exercises related to them. Again many of the exercises are very interesting and could be done as part of your project.
- In addition, any books on dynamical systems will have a part about chaos, so you will find other things in the library. Lots of things can be found in the web too!