The Homogeneous and Symmetric Space sine-Gordon models: a review

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WHAT THIS TALK IS ABOUT

- A brief introduction to the theories of interest $\Rightarrow$ the standard formulation of **non-abelian affine Toda field theories**
- The homogeneous and symmetric space sine-Gordon models $\Rightarrow$ the only NAAT-theories which are expected to have a sensible QFT counter-part, to be **integrable** at quantum level and to have a purely **massive spectrum**
- Results and open problems for both kinds of theories $\Rightarrow$ **unstable particles and parity breaking**

I. THE NON-ABELIAN TODA FIELD THEORIES

Let g be a complex semi-simple finite-dimensional Lie algebra and σ a finite order automorphism of g of order n. Then the automorphism induces a gradation of g of the form:

$$g = \bigoplus_{j \in \mathbb{Z}} g_{\bar{\jmath}} \qquad [g_{\bar{\jmath}}, g_{\bar{k}}] \subset g_{\overline{\jmath+k}} \qquad \bar{\jmath} = j \text{ mod } n,$$

where $\sigma(a) = e^{2\pi i j/n} a$ for $a \in g_{\bar{\jmath}}$. The field of the theory $h(x,t)$ takes values on the group G_0 associated to the Lie algebra $g_{\bar{0}}$ and the Toda equation also involves two diagonalizable elements $\Lambda_+ \in g_{\bar{k}}$ and $\Lambda_- \in g_{\overline{n-k}}$ for $k \geq 0$.

The action can be written as:

$$S[h] = S_{WZNW}[h] - \frac{m^2}{2\pi\beta^2} \int d^2x \langle \Lambda_+, h^{-1} \Lambda_- h \rangle,$$

for G_0 non-abelian the coupling constant $1/\beta^2 \in \mathbb{Z}^+$ in the quantum theory (level).

m is a mass scale.

$\langle , \rangle$ is an invariant, non-degenerate Killing form.

See the following concerning the Wess-Zumino-Novikov-Witten action:

W. Witten, CMP92 (1984) 455; V.G. Knizhnik and A. Zamolodchikov, NPB247 (1984) 83

From this action follow the NAAT-equations:

$$\partial_-(h^{-1}\partial_+ h) = -m^2[\Lambda_+, h^{-1}\Lambda_- h]$$

$$\partial_+(\partial_- h h^{-1}) = -m^2[h\Lambda_+ h^{-1}, \Lambda_-]$$

which admit a zero-curvature form that makes integrability manifest:

$$[\partial_+ + h^{-1}\partial_+ h + im\Lambda_+, \partial_- + imh^{-1}\Lambda_- h] = 0$$

$x_\pm = x \pm t$ are the light-cone coordinates.

A.N. Leznov and M.V. Saveliev, CMP89 (1983) 59

D. I. Olive, M. V. Saveliev, J.W.R. Underwood, PLB311 (1993) 117; J.W.R. Underwood, hep-th/9304156

L.A. Ferreira,J.L. Miramontes and J. Sánchez Guillén, NPB449 (1995) 631

II. WHAT HAPPENS AT QUANTUM LEVEL?: THE HSG- AND SSSG-MODELS

C. R. Fernández-Pousa, M. V. Gallas, T. J. Hollowood and J. L. Miramontes, NPB484 (1997) 609.

In this paper they

- asked themselves the question: which of the above theories have a real positive action and a mass gap?
- found that ensuring these properties implied strong restrictions on $h, \Lambda_\pm$ and σ
- consequently found that only two families of NAAT-theories were good candidates for integrable quantum field theories with purely massive spectrum
- they called these families the homogeneous and symmetric space sine-Gordon models

III. THEORIES WITH REAL AND POSITIVE ACTION

Kinetic term real an positive $\Rightarrow G_0$ has to be compact and the Killing form must be a compact real form, which implies $h^\dagger = h^{-1}$ and consistency with eqs. of motion implies also that $\Lambda_\pm^\dagger = \Lambda_\pm$ so that $\overline{2k} = 0$.

This singles out two families of models:

- $k = 0$ and $n \geq 1$. In this case $h \in G_0$ and $\Lambda_\pm \in g_{\bar{0}}$ and the theory can be described just in terms of $g_{\bar{0}}$ which is of the general form $u(1) \oplus \ldots \oplus u(1) \oplus g_{ss}$ where g_{ss} is semi-simple. The complex sine-Gordon model is the simplest case with $g = su(2)$.
- $k = 1$ and $n = 2$. In this case σ is an involution of g that induces the gradation $g = g_{\bar{0}} \oplus g_{\bar{1}}$, where $g_{\bar{0}}$ is a compact subalgebra of g. This decomposition satisfies the properties
$$[g_{\bar{0}}, g_{\bar{0}}] \subset g_{\bar{0}}, \quad [g_{\bar{1}}, g_{\bar{1}}] \subset g_{\bar{0}}, \quad [g_{\bar{1}}, g_{\bar{0}}] \subset g_{\bar{1}},$$
which implies that these theories are associated to symmetric spaces G/G_0.
- $k > 1$ and $n = 2k$. This can be included in the above class by introducing the involution $\hat{\sigma} = \sigma^k$ and $\hat{g} = g_{\bar{0}} \oplus g_{\bar{k}}$.

IV. THEORIES WITH A MASS GAP

The perturbing potential $V(h) = \langle \Lambda_+, h^{-1}\Lambda_- h\rangle$ has the following symmetry:

$$V(\alpha_- h \alpha_+) = V(h),$$

for $\alpha_\pm \in G_\pm$ and $g_\pm = \mathrm{Ker}(ad_{\Lambda_\pm}) \cap g_{\bar{0}}$. $V(h)$ has $G_- \times G_+$ left-right symmetry.

There are flat directions of the potential $\Rightarrow$ the corresponding QFT will not have a mass gap.

In order to avoid this we need to eliminate these flat directions by introducing the constraints:

$$P_+(h^\dagger \partial_+ h) = P_-(\partial_- h h^\dagger) = 0,$$

where $P_\pm$ are projection operators into the subalgebras $g_\pm$.

The idea [Q.-H. Park'94] is to gauge these symmetry transformations. It turns out that this can only be achieved if $\Lambda_\pm$ also satisfy $g_+ = g_- = g_0^0$ and g_0^0 is an abelian subalgebra. This implies introducing two gauge-fields $A_\pm \in g_0^0$ and substituting the $S_{WZNW}[h]$ by a the gauged action $S_{WZNW}[h, A_\pm]$ associated to the coset G_0/G_0^0. In the gauge $A_\pm = 0$ the equations of motion for these fields reduce to the constraints of no flat directions of the potential above.

V. THE FINAL CLASSIFICATION

The set of theories that posses both a real and positive action and a mass gap is very limited:

i) The homogeneous sine-Gordon models:

Reality condition: $h \in G_0$ and $\Lambda_\pm \in g_{\bar{0}}$.

Mass gap condition: g_0^0 is a Cartan subalgebra of $g_{\bar{0}}$

The HSG-models are associated to perturbations of the WZNW-model associated to cosets of the form $G_k/U(1)^{r_g}$ where G is a semi-simple compact Lie group of rank r_g.

A lot of work has been carried out concerned with quantum aspects of the HSG-models:

Proof of quantum integrability: C.R. Fernández-Pousa, M.V. Gallas, T.J. Hollowood and J.L. Miramontes, NPB499 [PM] (1997) 673.

Semiclassical spectrum: C.R. Fernández-Pousa and J.L. Miramontes, NPB518 [PM] (1998) 745.

Exact S-matrices: J.L. Miramontes and C.R. Fernández-Pousa, PLB472 (2000) 392.

TBA-analysis: O.C.A, A. Fring, K. Korff and J.L. Miramontes, NPB575 (2000) 535

Form factors: O.C.A.,A. Fring and C. Korff, PLB484 (2000) 167; O.C.A. and A. Fring, NPB604 (2001) 367; PRD63 (2001) 021701; PRD64 (2001) 085007...

ii) The Symmetric space sine-Gordon models:

Reality condition: $h \in G_0$, $\Lambda_\pm \in g_{\bar{1}}$. They are in one-to-one correspondence with the compact symmetric spaces G/G_0. The perturbation $\langle \Lambda_+, h^{-1}\Lambda_- h\rangle$ is a matrix element of the WZNW-field taken in the representation of G_0 provided by $[g_{\bar{0}}, g_{\bar{1}}] \subset g_{\bar{1}}$

Mass gap condition: Defining the rank of the symmetric space $\text{rank}(G/G_0)$ as the dimension of the largest abelian subalgebra of g contained in $g_{\bar{1}}$, we find that:

$$0 \leq \text{rank}(G) - \text{rank}(G/G_0) \leq p \leq \text{min}[\text{rank}(G_0), \text{rank}(G) - \nu]$$

where $p = \text{dim}(g_0^0)$ and $\nu = 2$ or 1 depending on whether or not $\Lambda_\pm$ are independent of each other, respectively.

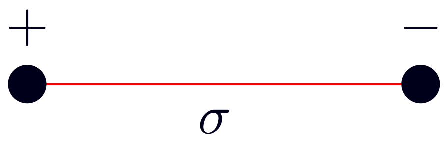
The SSSG-models are associated to perturbations of the WZNW-model associated to cosets of the form $(G_0)_k/U(1)^p$ where $g_0^0 = u(1)^p$. The value of p varies in the range above and depends on the choice of $\Lambda_\pm$. The case $\text{rank}(G) = \text{rank}(G/G_0)$ corresponds to the so-called split models. For $p = 0$ they have been studied in some detail (quantum integrability proven, classical mass spectrum found...).

O.C.A. and J.L. Miramontes, NPB581 (2000) 643; V. A. Brazhnikov, NPB501 (1997) 685

VI. THE EXACT S-MATRICES OF THE HSG-MODELS: The $SU(3)_2$-HSG model

J.L. Miramontes and C.R. Fernández-Pousa, PLB472 (2000) 392; O.C.A, A. Fring, K. Korff and J.L. Miramontes, NPB575 (2000) 535; O.C.A. and A. Fring, NPB604 (2001) 367

This is the simplest non-trivial model of this class. It consists of two self-conjugated particles $\pm$ which can be "attached" to the vertices of the A_2-Dynkin diagram:



The two particle S-matrices are and their structure is related to the Lie-algebras A_2 and A_1:

$$S_{\pm\pm}(\theta) = -1, \qquad S_{\pm\mp}(\theta) = \pm \tanh \frac{1}{2}(\theta \pm \sigma - \frac{i\pi}{2})$$

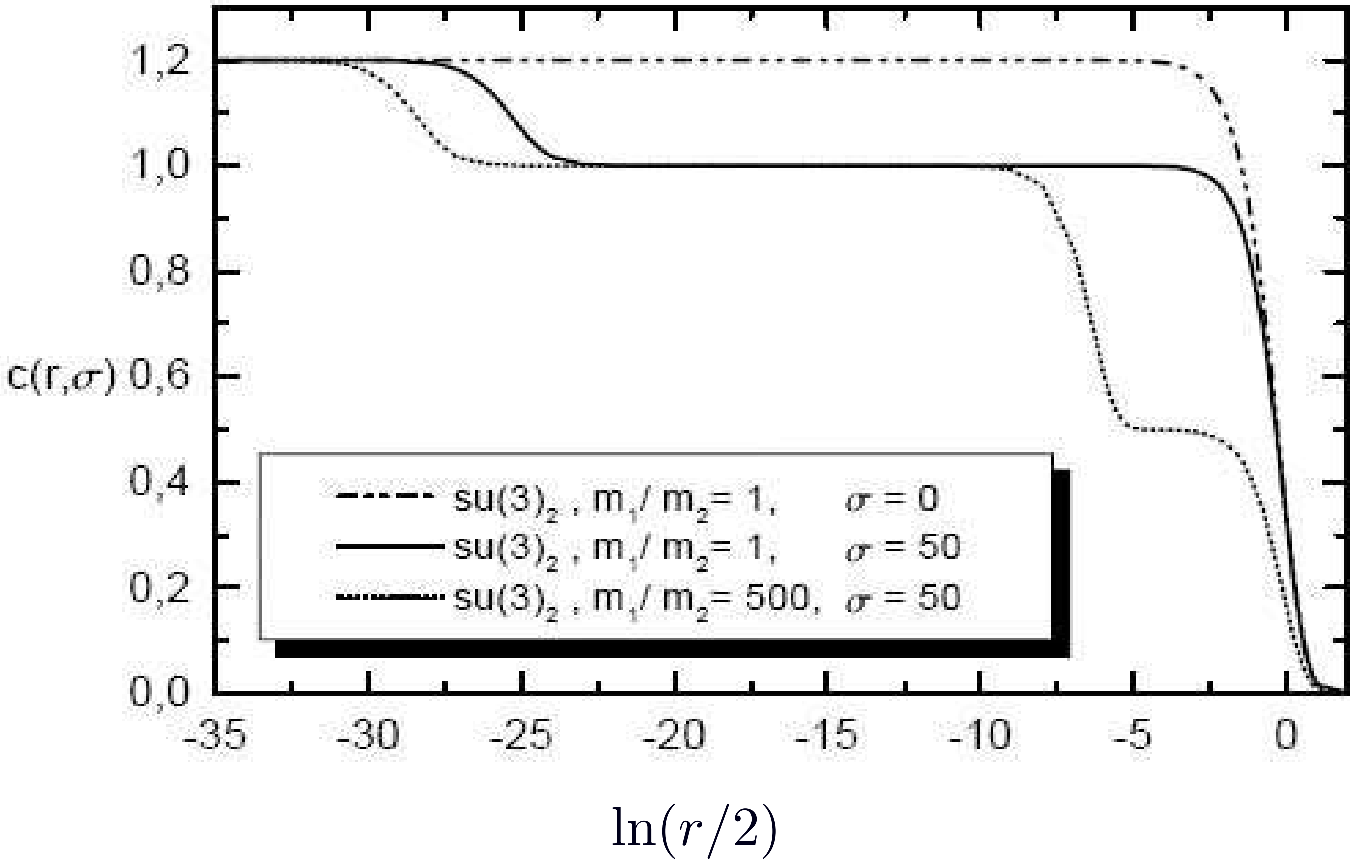
where $\sigma = \ln \sqrt{\frac{(\vec{\alpha}_1 \cdot \Lambda_+)(\vec{\alpha}_2 \cdot \Lambda_-)}{(\vec{\alpha}_1 \cdot \Lambda_-)(\vec{\alpha}_2 \cdot \Lambda_+)}}$ plays the role of a resonance parameter ($S_{\pm\mp}(\theta)$ has a pole at $\mp\sigma - i\pi/2$).

For $\sigma \to \infty$ the S-matrices $S_{\pm\mp} = 1$, so that the theory consists of two non-interacting copies of the Ising model. From the Dynkin diagram point of view it is like breaking the edge connecting the two vertices!

Several interesting features:

- The particles interact to form an unstable particle characterized by a pole of the S-matrix in the unphysical sheet. Using the Breit-Wigner formula one can show that the mass of this particle is $\sqrt{m_1 m_2} e^{|\sigma|/2}$ for σ large and m_1, m_2 the masses of the stable particles,
- the S-matrix breaks parity symmetry,
- despite its simplicity this theory is a perturbation of a WZNW-model associated to the coset $SU(3)/U(1)^2$ and level $k = 2$. This theory has $c = 6/5$ and the perturbing field has dimension $\Delta = 3/5$. The operator content of the theory is rather involved!

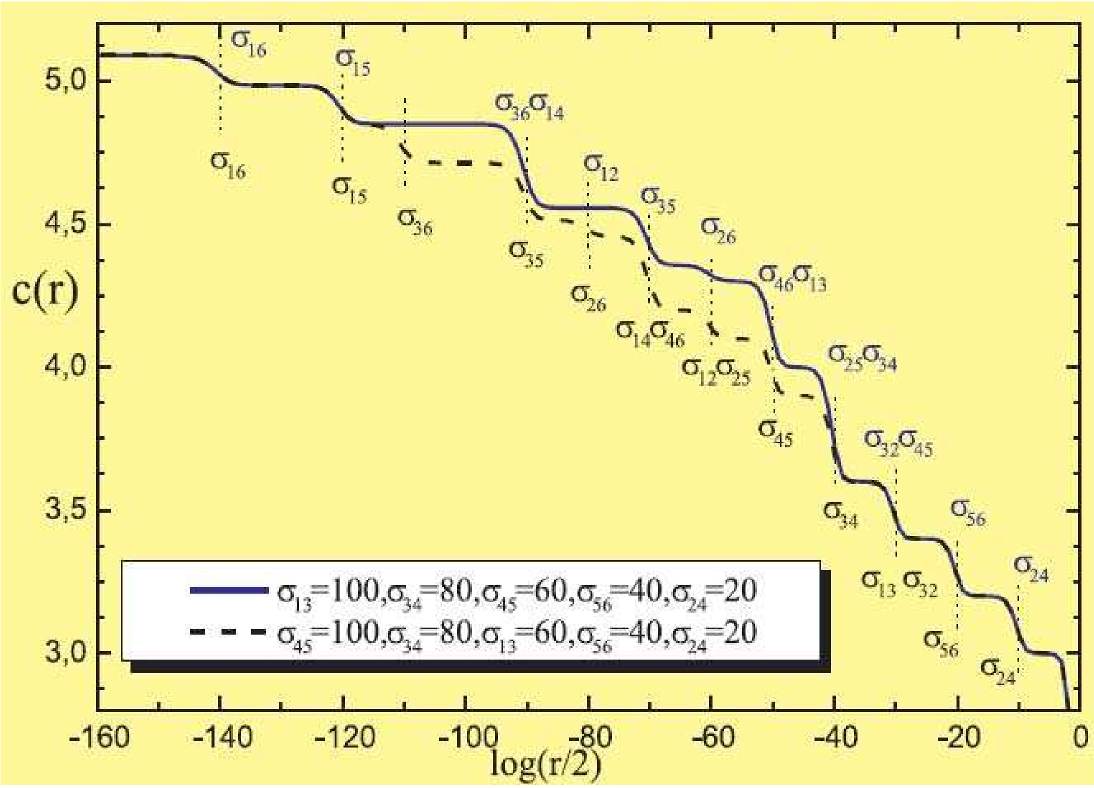
All the HSG-models can be regarded as r_g copies of A_{k-1}-minimal Toda theories (the mass spectrum of the stable particles and bound state structure is identical to that of minimal Toda! [R. Köberle and J. A. Swieca, PLB86 (1979) 209]). These theories then interact with each other by means of S-matrices including unstable particle poles. The structure of these S-matrices is dictated by that of the Lie algebra g.



O.C.A, A. Fring, K. Korff and J.L. Miramontes, NPB575 (2000) 535

The emergence of plateaux, as well as their width and height can be explained systematically thanks to the Lie algebraic structure of the models!

The particle spectrum complicates very much as the rank r increases e.g. for the $(E_6)_2$-HSG model:



O.C.A, J. Dreißig and A. Fring, EPJC35 (2004) 393; O.C.A. and A. Fring, Prog. Math. 237 (2005) 59.

P.E. Dorey and J.L. Miramontes NPB697 (2004) 405.

VII. SSSG-THEORIES AND SYMMETRIC SPACES

They are perturbations of WZNW-models associated to a compact group G_0 associated to an algebra $g_{\bar{0}}$ and $\Lambda_\pm \in g_{\bar{1}}$. The decomposition $g = g_{\bar{0}} \oplus g_{\bar{1}}$ together with the commutation relations that we have seen characterize a compact symmetric space G/G_0. In particular, the symmetric spaces of maximal rank are:

Type	G/G_0	rank(G/G_0)=rank(G)	dim(G/G_0)
A2	$SU(n)/SO(n)$	$n-1$	$\frac{1}{2}(n-1)(n+2)$
A1	$SO(2n)/SO(n)\times SO(n)$	n	n^2
A1	$SO(2n+1)/SO(n)\times SO(n+1)$	n	$n(n+1)$
B	$Sp(n)/U(n)$	n	$n(n+1)$
A2	$E_6/sp(4)$	6	42
A1	$E_7/su(8)$	7	70
A1	$E_8/so(16)$	8	128
A1	$F_4/Sp(3)\times SU(2)$	4	28
A1	$G_2/SU(2)\times SU(2)$	2	8

S. Helgasson, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, 1990

Here we concentrate on the case when g is simple (type I symmetric spaces)

S. Helgasson, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, 1990

V.G. Kac, Infinite dimensional Lie algebras, 3rd ed. Cambridge University Press,1990

Type I symmetric spaces can be further subdivided into three types [A1, A2, B], which correspond to the three types of solutions to the equation [Theorem 8.6 in Kac's book]:

$$2 = r \sum_{k=0}^{\ell} a_i s_i,$$

where $s_0, s_1, \ldots, s_\ell$ is a sequence of non-negative relatively prime integers and $a_0, \ldots, a_\ell$ are the Kac's labels corresponding to the Dynkin diagram of the (twisted if $r \neq 1$) Kac-Moody algebra $g^{(r)}$.

[A1] $r = 1, \quad a_k = 2s_k = 2$ for some k and $s_i = 0 \quad \forall i \neq k$

[A2] $r = 2, \quad a_k = s_k = 1$ for some k and $s_i = 0 \quad \forall i \neq k$

[B] $r = 1, \quad a_k = s_k = a_p = s_p = 1$ for some k, p and $s_i = 0 \quad \forall i \neq k, p$

This classification allows for an easy characterization of the subspaces $g_{\bar{0}}$ and $g_{\bar{1}}$ which we need in order to evaluate the conformal dimension of the perturbing field.

VIII. SSSG-THEORIES: CHARACTERIZATION OF SYMMETRIC SPACES

Theorem [Proposition 8.6 in Kac's book]:

(a) Let $i_1, \ldots, i_p$ be all indices for which $s_{i_1} = \ldots = s_{i_p} = 0$. Then $\bar{g}_0$ (the complexification of $g_{\bar{0}}$) is isomorphic to the direct sum of an $(\ell - p)$-dimensional centre and a semi-simple Lie algebra whose Dynkin diagram if a sub-diagram of the Dynkin diagram of $\bar{g}^{(r)}$ consisting of vertices $i_1, \ldots, i_p$.

(b) Let $j_1, \ldots, j_n$ be all indices for which $s_{j_1} = \ldots = s_{j_n} = 1$. Then the representation of $\bar{g}_0$ provided by $[\bar{g}_0, \bar{g}_1] \subset \bar{g}_{\bar{1}}$ is isomorphic to a direct sum of n irreducible modules of highest weights $-\alpha_{j_1}, \ldots, -\alpha_{j_n}$.

$$g_{\bar{0}} = \begin{cases} \bigoplus_{i=1}^{q} g^{(i)}, & \text{for type A1 and A2} \\ \bigoplus_{i=1}^{q} g^{(i)} \oplus u(1), & \text{for type B} \end{cases}$$

with $q = 1$ or 2 and $g^{(i)}$ compact and simple.

A1 and A2 theories are perturbations of WZNW models related to cosets of the form,

$$\bigotimes_{i=1}^{q} G_{k_i}^{(i)}/U(1)^p,$$

where k_i are the levels which are quantized according to

$$\frac{1}{\hbar\beta^2} = \frac{\vec{\Psi}^2_{g^{(i)}}}{2} k_i,$$

where $\vec{\Psi}^2_{g^{(i)}}$ is the square length of the long roots of $g^{(i)}$ with respect to the bilinear form of g_0.

$$c = \sum_{i=1}^{q} \frac{k_i \mathrm{dim}(g^{(i)})}{k_i + h_i^\vee} - p,$$

where $h_i^\vee$ is the dual Coxeter number of $g^{(i)}$.

The conformal dimension of the perturbing field is given by,

$$\Delta = \sum_{i=1}^{q} \frac{C_2(g^{(i)})/\vec{\Psi}^2_{g^{(i)}}}{k_i + h_i^{\vee}},$$

since the field lives in the highest weight representation of G_0, the quadratic Casimir is given by

$$C_2(g^{(i)}) = \langle \vec{\Lambda}, \vec{\Lambda} + 2\delta^{(i)} \rangle,$$

where $\vec{\Lambda} = -\vec{\alpha}_{j_1}$ (j_1 was the index for which $s_{j_1} = 1$) is the highest weight and $\delta^{(i)}$ is half the sum of the positive roots of $g^{(i)}$

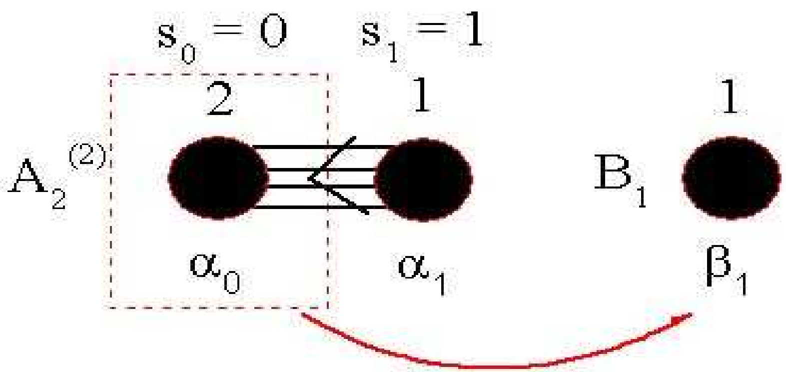
W. Witten, CMP92 (1984) 455

V.G. Knizhnik and A. Zamolodchikov, NPB247 (1984) 83

P. Goddard and D.I. Olive, IJMP1 (1986) 303

IX. A SIMPLE EXAMPLE: THE SYMMETRIC SPACE $SU(3)/SO(3)$

In this case $\bar{g} = A_2$ and $\bar{g}_0 = B_1$. The only way to have this is to choose $r = 2$ and $\vec{s} = (0, 1)$. The Dynkin diagrams are just



From the picture we see that the Dynkin diagram of $so(3)$ is a sub-diagram of $su(3)$ with $\frac{\vec{\Psi}^2_{A_2}}{\vec{\Psi}^2_{B_1}} = 4$. The highest weight $\vec{\Lambda} = -\vec{\alpha}_1 = 2\vec{\alpha}_0 = 2\vec{\beta}_1$ so $\frac{C_2(B_1)}{\vec{\Psi}^2_{B_1}} = \frac{\langle 2\vec{\beta}_1, 3\vec{\beta}_1 \rangle}{\langle \vec{\beta}_1, \vec{\beta}_1 \rangle} = 6$

$$\Delta = \frac{6}{k+2} \qquad \text{and} \qquad c = \frac{3k}{k+2}$$

I have chosen this example because this is in fact the only SSSG-model for which an attempt to compute the exact S-matrix has been made [V. A. Brazhnikov, NPB501 (1997) 685]. In fact only few S-matrix elements in the semi-classical approximation ($k \to \infty$) have been computed but they reveal important information which are consistent with some general features of these theories which we observed in [O.C.A. and J.L. Miramontes, NPB581 (2000) 643]:

- It was proven that the SSSG-theories related to the $SU(3)/SO(3)$ are classical and quantum integrable. We were able to prove that for all split models with $p = 0$ (perturbations of $(G_0)_k$-WZNW model)
- The semi-classical mass spectrum obtained by Brazhnikov for the $SO(3)_k$ SSSG theory reveals the presence of unstable particles in the spectrum. The analysis we performed for all split models shows that this is likely the be the case in general
- Brazhnikov noticed that at classical level the theory possesses soliton solutions similar to those found in the sine-Gordon model. However, due to the fact that the Homotopy group of $SO(3)$ is $\mathbb{Z}_2$ these solitons carry conserved charges living in $\mathbb{Z}_2$ rather than $\mathbb{Z}$.

X. MASS SPECTRUM OF $(G_0)_k$-SSSG models

In our work we show that for the split models the set of constant field configurations h_0 that minimize the potential are the solutions to

$$[\Lambda_+, h_0^\dagger \Lambda_- h_0] = 0 \quad \text{with} \quad (\vec{\alpha}\cdot\Lambda_+)(\vec{\alpha}\cdot h_0^\dagger\Lambda_- h_0) > 0,$$

for all positive roots $\vec{\alpha}$ of g. For the split models, since rank$(G) =$ rank(G/G_0), one can choose a set of generators t^α associated to each positive root $\vec{\alpha}$ of g and those constitute a basis for G_0. Then it is possible to show that the set of vacuum configurations is given by

$$\mathcal{M}_0 = \{1, e^{\pi\vec{\mu}\cdot\vec{t}} | \vec{\mu} = \sum_{i=1}^{r} \frac{2n_i\vec{\alpha}_i}{\vec{\alpha}_i^2}, n_i = 0, 1\}$$

Putting $h = h_0 e^{\phi t^\alpha}$ into the equations of motion one finds the mass spectrum

$$m_\alpha = 2m\sqrt{(\vec{\alpha}\cdot\Lambda_+)(\vec{\alpha}\cdot\Lambda_-)}$$

therefore, for each positive root $\vec{\alpha}$ there is a fundamental particle described by a real field $\phi(x,t)$ with the mass above.

One can easily see that $m_{\alpha+\beta} \geq m_\alpha + m_\beta$, which suggests that particles associated to composite roots might be unstable (proven only for $SU(3)/SO(3)$!)

In particular, if we set $h = e^{\phi t^\alpha/\vec{\alpha}^2}$ the Lagrangian becomes

$$\mathcal{L} = \frac{1}{4\pi\beta^2\vec{\alpha}^2}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + m_\alpha^2(\cos\phi - 1)\right),$$

which is the well-known sine-Gordon Lagrangian. The masses of the soliton and anti soliton are:

$$M_{s,\bar{s}}(\vec{\alpha}) = \frac{2m_\alpha}{\vec{\alpha}^2\pi\beta^2}$$

and there are n breathers $n < \frac{2}{\beta^2\vec{\alpha}^2}$ with masses:

$$M_a(\vec{\alpha}) = 2M_s(\vec{\alpha})\sin\left(\frac{\vec{\alpha}^2\pi\beta^2 a}{4}\right)$$

This suggests that the quantum version of these models might be a theory consisting of rank(G) copies of the sine-Gordon model which interact with each other by means of unstable particles in a way which is characterized by the structure of Lie algebra $g_{\bar{0}}$.

XI. SOME CONCLUSIONS

- The HSG- and SSSG-models are QFTs whose classical version are NAAT-theories. They can be regarded as perturbations of WZNW-models.

- The HSG-models are perturbations of the WZNW-model associated to the coset $G_k/U(1)^{r_g}$ for a certain level k. Their quantum integrability has been proven, the S-matrices are known exactly and their structure is intimately linked to the Lie algebraic structures of g and A_{k-1}. They have resonance poles associated to unstable particles and break parity invariance.

- The SSSG-models are perturbations of the WZNW-model associated to cosets $(G_0)_k/U(1)^p$ for a certain level k and different values of p. For $p = 0$ they are just perturbations of the $(G_0)_k$ WZNW-model and they belong to a sub-class of theories known as split models. Their quantum integrability has been proven and their classical mass spectrum studied.

- Although a lot of work needs to be done, our present knowledge supports the idea that the S-matrices of the latter theories will be closely related to those of the sine-Gordon model and that they will also include unstable particles in the spectrum.
- It would be nice to construct the exact S-matrices associated to the SSSG-theories. Once those are known many other interesting quantities could be computed, in particular form factors and correlation functions.
- The first step could be to extend the work of Brazhnikov for the symmetric space $SU(3)/SO(3)$. A full understanding of this case should shed light on the general structure.

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THANK YOU VERY MUCH!

If you are interested, you will find this talk at:

http://www.staff.city.ac.uk/o.castro-alvaredo/cv/imperial.pdf