Bi-partite entanglement entropy from branch point twist fields in integrable quantum field theory

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- How? $\Rightarrow$ replica trick and integrability
- Bi-partite entanglement entropy $\Leftrightarrow$ correlation functions of branch point twist fields

Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.
  A typical example: a pair of opposite-spin electrons:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right) \;, \quad \langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

- What is special: Bell's inequality says that this cannot be described by **local variables**.
- A situation that looks similar to $|\psi\rangle$ but without entanglement is (a factorizable state):

$$|\hat{\psi}\rangle = \frac{1}{2} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right) = \frac{1}{2} \left(|\uparrow\rangle + |\downarrow\rangle \right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle \right)$$

- This is particular to **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \;, \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

(for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).

How and why to measure (or quantify) quantum entanglement?

- Measuring quantum entanglement is useful: as such measurement may have applications to the design of (still theoretical) quantum computers. It is also a fundamental property of quantum systems.

- In **pure states**, there are various proposals to measure quantum entanglement. Consider the **entanglement entropy** and let us look at a more complicated system:
  - With the Hilbert space a tensor product $\mathcal{H} = s_1 \otimes s_2 \otimes \cdots \otimes s_N = A \otimes \bar{A}$, and a given state $|\mathrm{gs}\rangle \in \mathcal{H}$, calculate the **reduced density matrix**:

$$\rho_A = \mathrm{Tr}_{\bar{A}}(|\mathrm{gs}\rangle\langle\mathrm{gs}|)$$

$$\cdots s_{i-1} \otimes \underbrace{s_i \otimes s_{i+1} \otimes \quad \cdots \quad \otimes s_{i+L-1}}_{A} \otimes s_{i+L} \quad \cdots$$

  - The entanglement entropy is the resulting **von Neumann entropy**:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\sum_{\substack{\text{eigenvalues of } \rho_A \\ \lambda \neq 0}} \lambda \log(\lambda)$$

Interpretation of the entanglement entropy

- It is the entropy that is measured in a subsystem A, once the rest of the system $\bar{A}$ – the environment – is forgotten.

  If we think A is all there is, we will think the system is in a mixed state, with density matrix given by ρ_A. The entropy of ρ_A measures how mixed ρ_A is. This mixing is due to the connections, or entanglement, with the environment.
- It was proposed as a way to understand black hole entropy [Bombelli, Koul, Lee, Sorkin 1986].
- Then it was proposed as a measure of entanglement [Bennet, Bernstein, Popescu, Schumacher 1996].
- Examples:
  - Tensor product state:
    $|\mathrm{gs}\rangle = |A\rangle \otimes |\bar{A}\rangle \Rightarrow \rho_A = |A\rangle\langle A| \Rightarrow S_A = -1\log(1) = 0$.
  - The maximally entangled state $|\mathrm{gs}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$:
    $\rho_{1^{\mathrm{st}}\ \mathrm{spin}} = \dfrac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) \Rightarrow S_{1^{\mathrm{st}}\ \mathrm{spin}} = -2\times\Big(-\log(\sqrt{2})\Big) = \log(2)$
- In general the entanglement entropy is not "directional", that is $S_A = S_{\overline{A}}$.

Scaling limit

- Say $|\mathrm{gs}\rangle$ is a ground state of some local spin-chain Hamiltonian, and that the chain is infinitely long.
- An important property of $|\mathrm{gs}\rangle$ is the **correlation length** ξ:

$$\langle \mathrm{gs}|\hat{\sigma}_i\hat{\sigma}_j|\mathrm{gs}\rangle \sim e^{-|i-j|/\xi} \text{ as } |i-j| \to \infty$$

- If there are parameters in the Hamiltonian (e.g. temperature, magnetic field etc) such that for certain values $\xi \to \infty$. This is a **quantum critical point**.
- We may adjust these parameters in such a way that the length L of A is proportional to ξ: $L/\xi = mr$.
- This is the **scaling limit**, and what we obtain is a **quantum field theory**. m here is a mass scale – we may have many masses m_α associated to many correlation lengths – and r is the dimensionful length of region A in the scaling limit.
- The resulting entanglement entropy has a **universal** part: a part that does not depend very much on the details of the Hamiltonian.

Short- and large-distance entanglement entropy

Consider $\varepsilon = 1/(m_1\xi)$, a non-universal QFT cutoff with dimenions of length. Then:

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior [Holzhey, Larsen, Wilczek 1994; Calabrese, Cardy 2004]

$$S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$S_A \sim -\frac{c}{3}\log(m_1\varepsilon) + U$$

where c is the central charge of the corresponding critical point. One of the main results of our work was to find the next-to-leading order correction to this behavior.

Partition functions on multi-sheeted Riemann surfaces

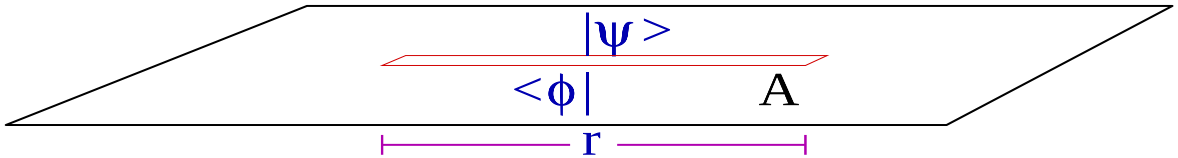
[Callan, Wilczek 1994; Holzhey, Larsen, Wilczek 1994]

- We can use the “replica trick” for evaluating the entanglement entropy:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_A(\rho_A^n)$$

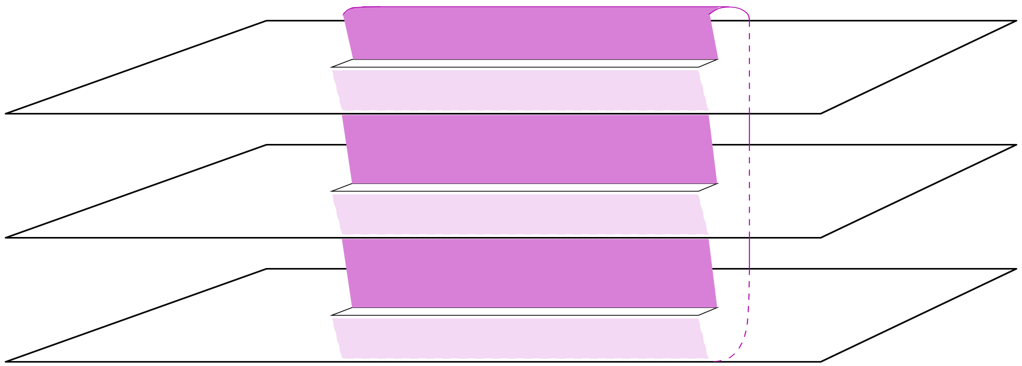
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface ($\mathrm{Tr}_A(\rho_A)$ is the partition function of the original theory!):

$${}_A\langle\phi|\rho_A|\psi\rangle_A \quad \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \quad \sim \quad Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n \quad :$$



Branch-point twist fields

[J.L. Cardy, O.C.A, B. Doyon 2007]

- Consider many copies of the QFT model on the usual $\mathbb{R}^2$:

$$\mathcal{L}^{(n)}[\varphi_1, \ldots, \varphi_n](x) = \mathcal{L}[\varphi_1](x) + \ldots + \mathcal{L}[\varphi_n](x)$$

- There is an obvious symmetry under cyclic exchange of the copies:

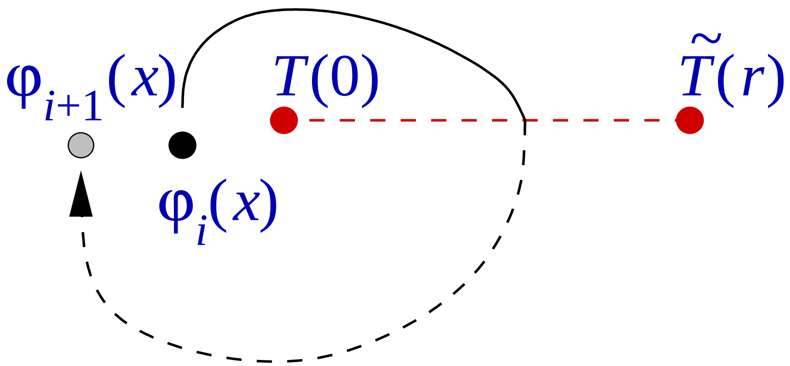
$$\mathcal{L}^{(n)}[\sigma\varphi_1, \ldots, \sigma\varphi_n] = \mathcal{L}^{(n)}[\varphi_1, \ldots, \varphi_n]\,, \quad \text{with} \quad \sigma\varphi_i = \varphi_{i+1 \bmod n}$$

- Whenever we have a symmetry in a QFT we can associate a field to it. We will call the fields associated to the $\mathbb{Z}_n$ symmetry introduced here **twist fields**.

- Another twist field $\tilde{\mathcal{T}}$ is associated to the inverse symmetry σ^{-1}, and we have

$$
\begin{aligned}
\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} &\propto \int_{C_{0,r}} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} d^2x\, \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x)\right] \\
&= Z_n
\end{aligned}
$$

$C_{0,r}$:

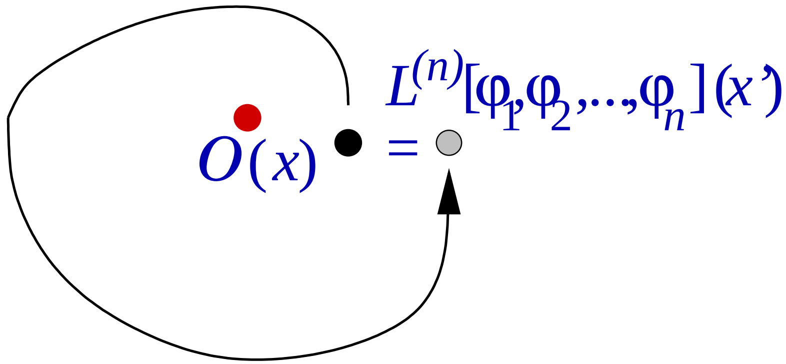


Locality in QFT

- A field $\mathcal{O}(x)$ is **local** in QFT if measurements associated to this field are quantum mechanically independent from measurements of the **energy density** (or **Lagrangian density**) at space-like distances. That is, equal-time commutation relations vanish:

$$[\mathcal{O}(\mathrm{x}, t=0), \mathcal{L}^{(n)}(\mathrm{x}', t=0)] = 0 \quad (\mathrm{x} \neq \mathrm{x}')\ .$$

- This means that:



- **Branch-point twist fields are local fields in the n-copy theory.**

Short- and large-distance entanglement entropy revisited

Hence we have

$$Z_n = D_n \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

where D_n is a normalisation constant, and d_n is the scaling dimension of $\mathcal{T}$ [Calabrese, Cardy 2004]:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim r^{-2d_n} \ \Rightarrow \ S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim \langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}} \ \Rightarrow \ S_A \sim -\frac{c}{3}\log(m_1\varepsilon) + U$$

Form factors and two-point functions in integrable models

- In order to simplify matters let us now think of a QFT with a single particle spectrum. In the n-replica model $\mathcal{L}^{(n)}$, there will be n particles that we can label by $j = 1, \ldots, n$.

- The two-point function of branch-point twist fields can be decomposed as follows, giving a **large-distance expansion**:

$$\begin{aligned} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \mathrm{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\ &= \sum_{\text{state } k} \langle \mathrm{gs}|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \end{aligned}$$

  where $\sum_k |k\rangle\langle k|$ is a sum over a complete set of states in the Hilbert space of the theory.

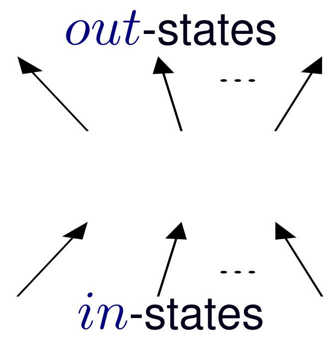
- The matrix elements $\langle \mathrm{gs}|\mathcal{T}(0)|k\rangle$ are called **form factors**.

- For integrable models, an specific program exists (**form factor program**) that allows their exact (non-perturbative) computation.

- However the program needs to be modified to include twist fields correctly.

Asymptotic states

- In QFT, the Hilbert space is described by particles coming from the far past (in-states) or going to the far future (out-states). The overlap between in- and out-states is the **scattering matrix**.



- With particle trajectories chosen to meet all at a point in space-time, the set of all possible configurations of incoming particles (particle types and rapidities) forms a basis for the Hilbert space. Likewise for outgoing particles.

- These in-states or out-states are denoted $|\theta_1, \theta_2, \ldots, \theta_k\rangle^{in,out}_{\mu_1,\mu_2,\ldots,\mu_k}$ with $\theta_1 > \ldots > \theta_k$ for in-states and the opposite for out-states, where θ_i's are rapidities and μ_i's are particle types. Here we assume all particles of the model to be massive.

- Energy and momentum of these states are the sums of those of individual particles: $E = \sum_{i=0}^{k} m_{\mu_i} \cosh \theta_i$ and $P = \sum_{i=0}^{k} m_{\mu_i} \sinh \theta_i$.

- In terms of these states, the generic state $|k\rangle = |\theta_1, \theta_2, \ldots, \theta_k\rangle^{in}_{\mu_1 \ldots \mu_k}$.

The two-point function (again)

- The two-point function of branch-point twist fields can be decomposed into the in-basis, giving a **large-distance expansion**:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} = \langle \mathrm{vac}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{vac}\rangle =$$
$$\sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k=1}^{n}\int \frac{d\theta_1\cdots d\theta_k}{(2\pi)^k}\,|F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)|^2 e^{-r\sum_{i=1}^{k} m_{\mu_i}\cosh\theta_i}$$

where

$$F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k) = \langle \mathrm{vac}|\mathcal{T}(0)|\theta_1,\ldots,\theta_k\rangle^{in}_{\mu_1,\ldots,\mu_k}$$

are the k-particle form factors of the twist-field $\mathcal{T}$.

- Typically the expansion is rapidly convergent as the number of particles is increased (the lower-particle form factors will give the main contribution)

Form factors of branch-point twist fields

[P. Weisz 1977; M. Karowski, P. Weisz 1978; F.A. Smirnov 1992]

[J.L. Cardy, O.C.A,B. Doyon 2007]

- Consider an integrable QFT with one particle, no bound state, and S-matrix $S(\theta)$
- The scattering matrix of the n-replica theory is:

$$S_{ij}(\theta) \quad = \quad S(\theta)^{\delta_{ij}}, \quad \text{with} \quad \delta_{ij} = \begin{cases} 0, & \text{for} \quad i \neq j \\ 1, & \text{for} \quad i = j \end{cases}$$

$$F_k^{\cdots\mu_i\mu_{i+1}\cdots}(\ldots,\theta_i,\theta_{i+1},\ldots) = S_{\mu_i\mu_{i+1}}(\theta_i-\theta_{i+1})F_k^{\cdots\mu_{i+1}\mu_i\cdots}(\ldots,\theta_{i+1},\theta_i,\ldots)$$

$$F_k^{\mu_1\mu_2\cdots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = F_k^{\mu_2\cdots\mu_k\ \mu_1+1}(\theta_2,\ldots,\theta_k,\theta_1)$$

$$-i\text{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mu\mu\mu_1\cdots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1\ldots,\theta_k) = F_k^{\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k)$$

$$-i\text{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mu\ \mu+1\ \mu_1\cdots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1\ldots,\theta_k) = -\prod_{i=1}^{k} S_{\mu\mu_i}(\theta_{0i})F_k^{\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k)$$

- These equations can be solved recursively by relating lower- to higher-particle form factors.

Large distance corrections

[J.L. Cardy, O.C.A., B. Doyon 2007], [O.C.A, B. Doyon 2008], [B. Doyon 2008]

Our result: for any integrable QFT, the entropy with its **first correction to saturation** at large distances is:

$$\boxed{S_A \sim -\frac{c}{3}\log(m\varepsilon) - U - \frac{1}{8}K_0(2rm) + O\left(e^{-3rm}\right)}$$

where m is the mass of the particle.

- The next-to-leading order correction term depends only on the particle spectrum, but not on the interaction between particles (i.e. not the S-matrix).
- In our work we have extended this result to any integrable QFT.
- In our last work [O.C.A, B. Doyon 2008] we have also computed all the **remaining higher order corrections** for the special case of a free Fermion theory and obtained also all corrections for a free Fermion theory with a boundary.

The two-particle contribution (next-to-leading order correction to the entropy)

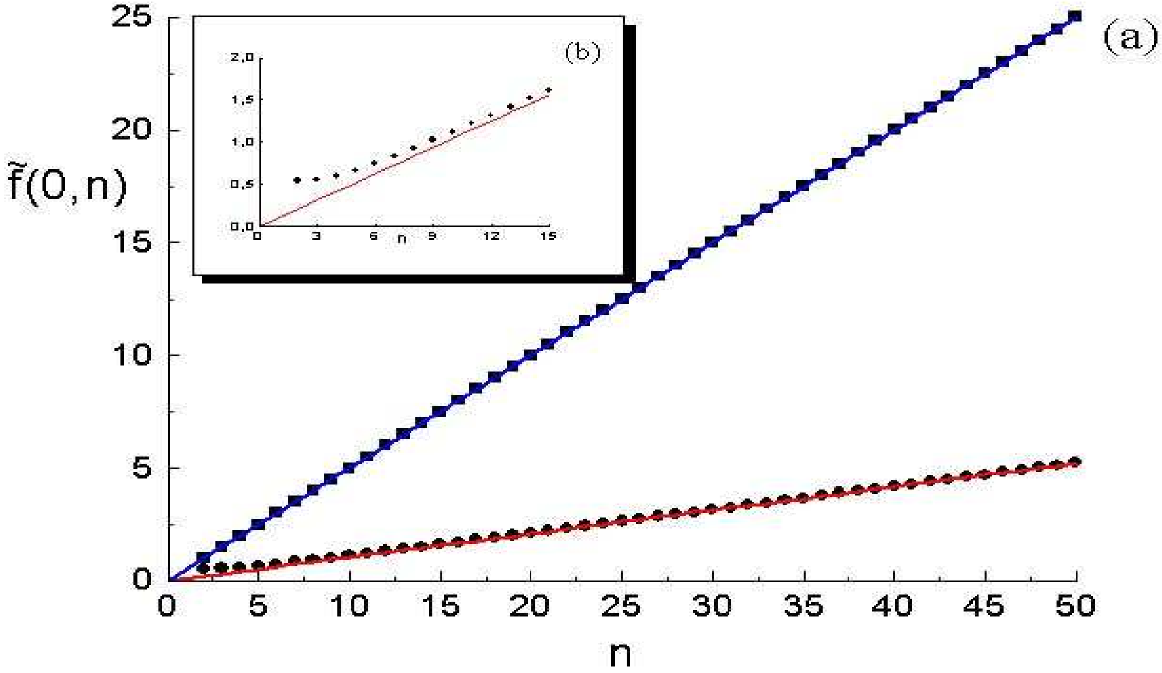
$$\begin{aligned}
\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \mathrm{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\
&= \sum_{\text{state } k} \langle \mathrm{gs}|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\
&= \langle \mathcal{T}\rangle^2 + n\sum_{j=1}^{n}\int d\theta_1 d\theta_2 e^{-mr(\cosh\theta_1+\cosh\theta_2)}|F_2^{1j}(\theta_1-\theta_2)|^2 + \ldots \\
&= \langle \mathcal{T}\rangle^2\left(1+\frac{n}{4\pi^2}\int_{-\infty}^{\infty} f(\theta,n)K_0(2mr\cosh(\theta/2)d\theta + \ldots\right)
\end{aligned}$$

where

$$f(\theta,n) = \langle \mathcal{T}\rangle^{-2}\sum_{j=0}^{n-1}|F_2^{11}(-\theta+2\pi i j)|^2$$

- Here we are considering a theory with vanishing one-particle form factor (even if it was non-vanishing it would not change the result for the entropy).
- Main difficulty: analytically continue $f(\theta,n)$ for $n\in\mathbb{R}$, $n\leq 1$, then take the derivative at $n=1$.

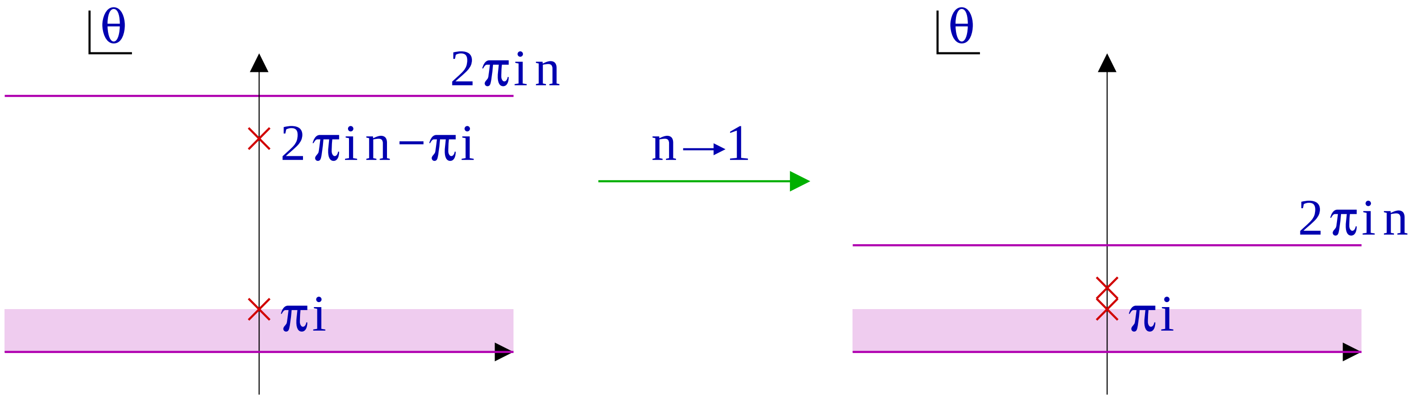
In order to compute the entropy we would like to evaluate $\lim_{n\to 1}\frac{d}{dn}(nf(\theta,n)) \Rightarrow$ analytic continuation $\tilde{f}(\theta,n)$ of $f(\theta,n)$ from $n=1,2,3,\ldots$ to $n\in[1,\infty)$



The analytic continuation $\tilde{f}(\theta,n)$ of $f(\theta,n)$ does not converge uniformly as $n\to 1$ on $\theta\in\mathbb{R}$, that is, $\tilde{f}(0,1)\neq f(0,1)=0$

The analytic continuation

The non-zero value of $\tilde{f}(0,1)$ is due to the collision of poles of $|F_2^{11}(2\pi ij)|^2$ as function of j as $n \longrightarrow 1$



Defining

$$s(\theta,j) = |F_2^{11}(-\theta + 2\pi ij)|^2 \quad \text{with} \quad f(\theta,n) = \langle \mathcal{T} \rangle^{-2} \sum_{j=1}^{n} s(\theta,j)$$

Extracting the poles:

$$s(\theta, j) \sim \frac{iF_2^{11}(-2\theta + 2\pi i n - i\pi)}{-\theta - 2\pi i j + 2\pi i n - i\pi} - \frac{iF_2^{11}(-2\theta + i\pi)}{-\theta - 2\pi i j + i\pi} + \mathrm{c.c.}$$

and re-summing them exactly gives

$$\tilde{f}(\theta, n) \sim \tilde{f}(0,1)\left(\frac{i\pi(n-1)}{2(\theta + i\pi(n-1))} - \frac{i\pi(n-1)}{2(\theta - i\pi(n-1))}\right), \quad \tilde{f}(0,1) = \frac{1}{2}$$

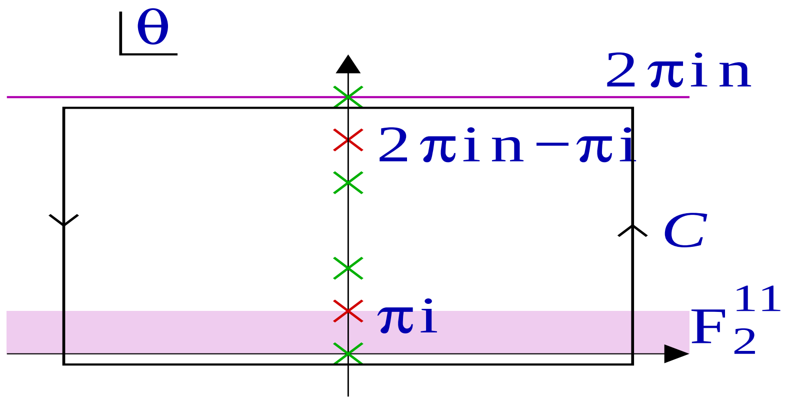
Hence the derivative is supported at $\theta = 0$:

$$\left(\frac{\partial}{\partial n}\tilde{f}(\theta, n)\right)_{n=1} = \pi^2 \tilde{f}(0,1)\delta(\theta)$$

There is an exact analytic continuation:

Consider the closed-contour integral

$$\int_{\mathcal{C}} \frac{dj}{2\pi i} \pi \cot \pi j \; F_2^{11}(2\pi i j)^2$$



Assuming $F_2^{11}(0) = 0$ and $F_2^{11}(\theta) = 0$ at $|\theta| \to \infty$:

$$\tilde{f}(0,n) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Im}(S(-\theta)) \coth\left(\frac{\theta}{2}\right) |F_2^{11}(\theta)|^2 d\theta$$

For the higher order corrections the analytic continuation is much harder to obtain. So far, we have managed to do it for the free Fermion theory (the only model for which all form factors are known) [O.C.A, B. Doyon 2008]

Conclusions and outlook

- The main result of our work so far is the derivation of the first correction to saturation of the entanglement entropy in any IQFT and the computation of all corrections for the Free fermion model (with and without boundaries).
- The key ingredients for this have been the introduction of branch point twist fields in terms of whose two-point function the entropy can be evaluated. The form factor program has been generalised to accommodate branch point twist fields.
- There is scope for generalization:
  - Extension of many of our results to interacting theories
  - Multi-partite entanglement entropy
  - Entanglement entropy in integrable QFT at finite temperature