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Entanglement entropy in 1+1 dimensional integrable QFTs: some ideas on the entropy of disconnected regions

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ICFT10, University of Kent, 15-16 April 2010

ISSN 1751-8113

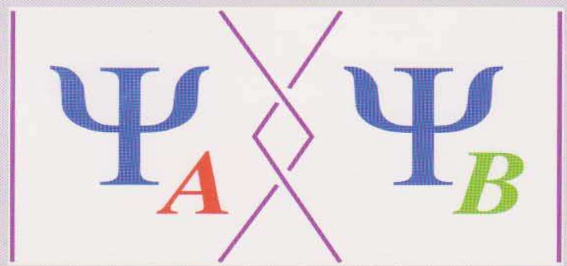
Journal of Physics A
Mathematical and Theoretical

Volume 42 Number 50 18 December 2009

Special issue

Entanglement entropy in extended quantum systems

Guest Editors: Pasquale Calabrese, John Cardy and Benjamin Doyon



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Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.

  A typical example: a pair of opposite-spin electrons:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \;, \quad \langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

- What is special: Bell's inequality says that this cannot be described by **local variables**.

- A situation that looks similar to $|\psi\rangle$ but without entanglement is a factorizable state:

$$|\hat{\psi}\rangle = \frac{1}{2} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) = \frac{1}{2} (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle)$$

- This is particular to **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \;, \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

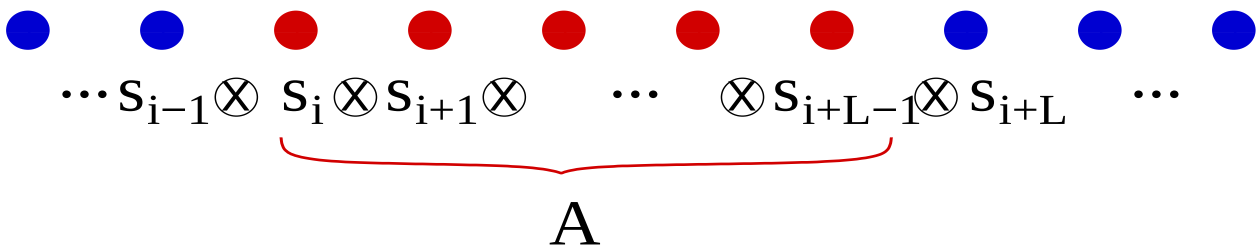
  (for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).

How and why to measure (or quantify) quantum entanglement?

- Measuring quantum entanglement is useful: as such measurement may have applications to the design of (still theoretical) quantum computers. It is also a fundamental property of quantum systems.

- In **pure states**, there are various proposals to measure quantum entanglement. Consider the **entanglement entropy** and let us look at a more complicated system:
  - With the Hilbert space a tensor product $\mathcal{H} = s_1 \otimes s_2 \otimes \cdots \otimes s_N = A \otimes \bar{A}$, and a given state $|\mathrm{gs}\rangle \in \mathcal{H}$, calculate the **reduced density matrix**:

$$\rho_A = \mathrm{Tr}_{\bar{A}}(|\mathrm{gs}\rangle\langle\mathrm{gs}|)$$



  - The entanglement entropy is the resulting **von Neumann entropy**:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\sum_{\substack{\text{eigenvalues of } \rho_A \\ \lambda \neq 0}} \lambda \log(\lambda)$$

Interpretation of the entanglement entropy

- It is the entropy that is measured in a subsystem A, once the rest of the system $\bar{A}$ – the environment – is forgotten.

  If we think A is all there is, we will think the system is in a mixed state, with density matrix given by ρ_A. The entropy of ρ_A measures how mixed ρ_A is. This mixing is due to the connections, or entanglement, with the environment.

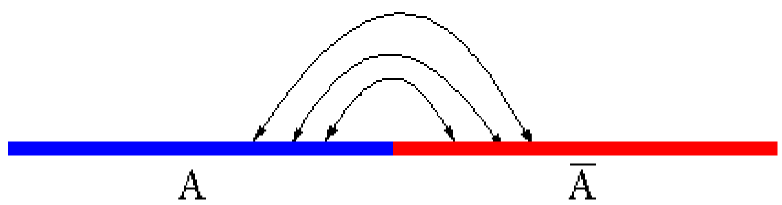
- It was proposed as a way to understand black hole entropy [Bombelli, Koul, Lee, Sorkin 1986].
- Then it was proposed as a measure of entanglement [Bennet, Bernstein, Popescu, Schumacher 1996].
- Examples:
  - Tensor product state:

    $|\mathrm{gs}\rangle = |A\rangle \otimes |\bar{A}\rangle \Rightarrow \rho_A = |A\rangle\langle A| \Rightarrow S_A = -1\log(1) = 0$.

- The maximally entangled state $|\mathrm{gs}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$:

$$\rho_{1^{\mathrm{st}}\ \mathrm{spin}} = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) \Rightarrow S_{1^{\mathrm{st}}\ \mathrm{spin}} = -2\times\left(-\log(\sqrt{2})\right) = \log(2)$$

- In general the entanglement entropy is not "directional", that is $S_A = S_{\overline{A}}$.



- We can think of the entropy as counting the number of "connections" between A and $\bar{A}$.

Scaling limit

- Say $|\text{gs}\rangle$ is a ground state of some local spin-chain Hamiltonian, and that the chain is infinitely long.
- An important property of $|\text{gs}\rangle$ is the **correlation length** ξ:

$$\langle \text{gs}|\hat{\sigma}_i\hat{\sigma}_j|\text{gs}\rangle \sim e^{-|i-j|/\xi} \text{ as } |i-j| \to \infty$$

- If there are parameters in the Hamiltonian (e.g. temperature, magnetic field etc) such that for certain values $\xi \to \infty$. This is a **quantum critical point**.
- We may adjust these parameters in such a way that $L, \xi \to \infty$ in such as way that the length L of A is proportional to ξ: $L/\xi = mr$.
- This is the **scaling limit**, and what we obtain is a **quantum field theory**. m here is a mass scale – we may have many masses m_α associated to many correlation lengths – and r is the dimensionful length of region A in the scaling limit.
- The resulting entanglement entropy has a **universal** part: a part that does not depend very much on the details of the Hamiltonian.

Short- and large-distance entanglement entropy

Consider $\varepsilon = 1/(m_1\xi)$, a non-universal QFT cutoff with dimenions of length. Then:

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior [Holzhey, Larsen, Wilczek 1994; Calabrese, Cardy 2004]

$$S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$S_A \sim -\frac{c}{3}\log(m_1\varepsilon) + U$$

where c is the central charge of the corresponding critical point. One of the main results of our work was to find the next-to-leading order correction to this behavior.

Large distance corrections

[J.L. Cardy, O.C.A., B. Doyon 2007], [O.C.A, B. Doyon 2008], [B. Doyon 2008]

Our result: for any integrable QFT, the entropy with its **first correction to saturation** at large distances is:

$$S_A \sim -\frac{c}{3}\log(m\varepsilon) + U - \frac{1}{8}K_0(2rm) + O\left(e^{-3rm}\right)$$

where m is the mass of the particle.

- The next-to-leading order correction term depends only on the particle spectrum, but not on the interaction between particles (i.e. not the S-matrix).
- In our work we have extended this result to any integrable QFT (even non-integrable [B. Doyon 2008]).
- In a recent work [O.C.A, B. Doyon 2008] we have also computed all the **remaining higher order corrections** for the special case of the Ising field theory with and without a boundary.

Partition functions on multi-sheeted Riemann surfaces

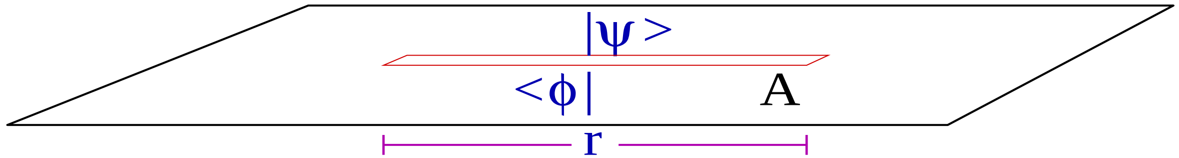
[Callan, Wilczek 1994; Holzhey, Larsen, Wilczek 1994]

- We can use the "replica trick" for evaluating the entanglement entropy:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1} \frac{d}{dn} \mathrm{Tr}_A(\rho_A^n)$$

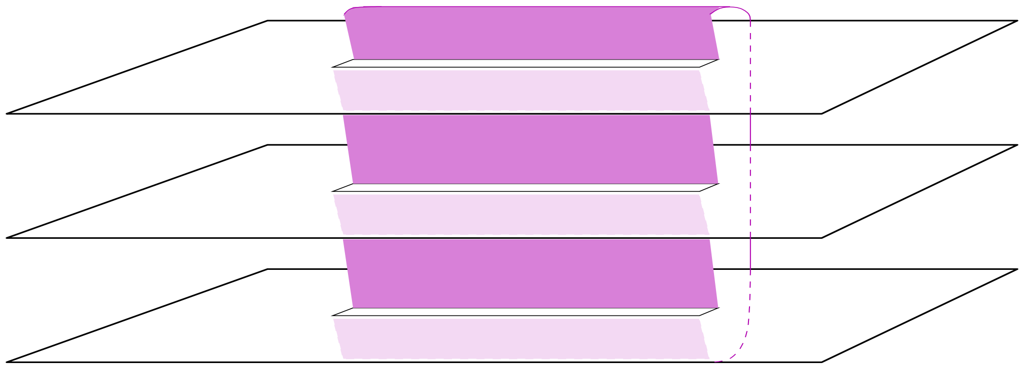
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface:

$${}_A\langle\phi|\rho_A|\psi\rangle_A \quad \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \quad \sim \quad Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n \quad :$$



Branch-point twist fields

[J.L. Cardy, O.C.A, B. Doyon 2007]

- Consider many copies of the QFT model on the usual $\mathbb{R}^2$:

$$\mathcal{L}^{(n)}[\varphi_1, \dots, \varphi_n](x) = \mathcal{L}[\varphi_1](x) + \dots + \mathcal{L}[\varphi_n](x)$$

- There is an obvious symmetry under cyclic exchange of the copies:

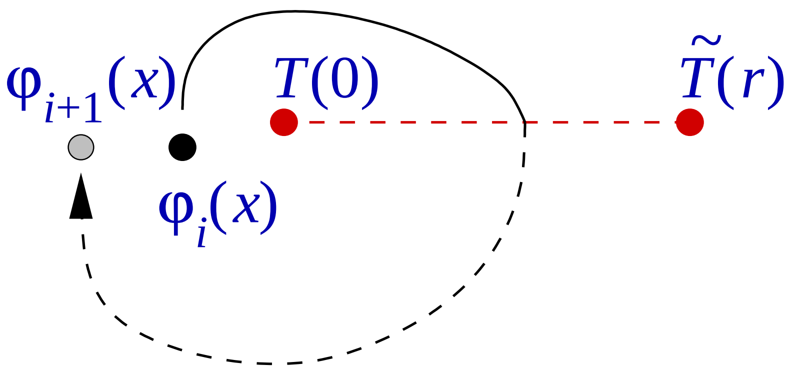
$$\mathcal{L}^{(n)}[\sigma\varphi_1, \dots, \sigma\varphi_n] = \mathcal{L}^{(n)}[\varphi_1, \dots, \varphi_n]\,, \quad \text{with} \quad \sigma\varphi_i = \varphi_{i+1 \bmod n}$$

- Whenever we have an internal symmetry in a QFT we can associate a twist field to it. We will call the fields associated to the $\mathbb{Z}_n$ symmetry introduced here **branch point twist fields**.

- Another twist field $\tilde{\mathcal{T}}$ is associated to the inverse symmetry σ^{-1}, and we have

$$\begin{aligned}\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} &\propto \int_{C_{0,r}} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} d^2x\ \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x)\right] \\ &= Z_n = \mathrm{Tr}_A(\rho_A^n)\end{aligned}$$

$C_{0,r}$:

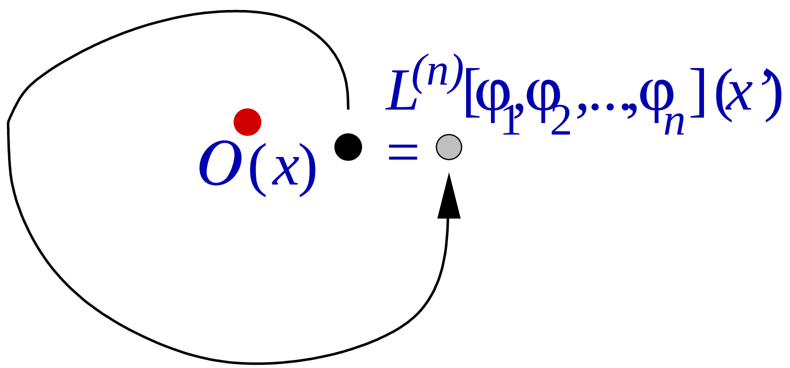


Locality in QFT

- A field $\mathcal{O}(x)$ is **local** in QFT if measurements associated to this field are quantum mechanically independent from measurements of the **energy density** (or **Lagrangian density**) at space-like distances. That is, equal-time commutation relations vanish:

$$[\mathcal{O}(\mathrm{x}, t=0), \mathcal{L}^{(n)}(\mathrm{x}', t=0)] = 0 \quad (\mathrm{x} \neq \mathrm{x}') \ .$$

- This means that:



- **Branch-point twist fields are local fields in the n-copy theory.**

Short- and large-distance entanglement entropy revisited

Hence we have

$$Z_n = D_n \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

where D_n is a normalisation constant, and d_n is the scaling dimension of $\mathcal{T}$ [Calabrese, Cardy 2004]:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim r^{-2d_n} \ \Rightarrow \ S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim \langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}} \ \Rightarrow \ S_A \sim -\frac{c}{3}\log(m_1\varepsilon) + U$$

Form factors and two-point functions in integrable models

- The two-point function of branch-point twist fields can be decomposed as follows, giving a **large-distance expansion**:

$$\begin{aligned}\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \mathrm{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\ &= \sum_{\text{state } k} \langle \mathrm{gs}|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle\end{aligned}$$

where $\sum_k |k\rangle\langle k|$ is a sum over a complete set of states in the Hilbert space of the theory.

- The matrix elements $\langle \mathrm{gs}|\mathcal{T}(0)|k\rangle$ are called **form factors**.

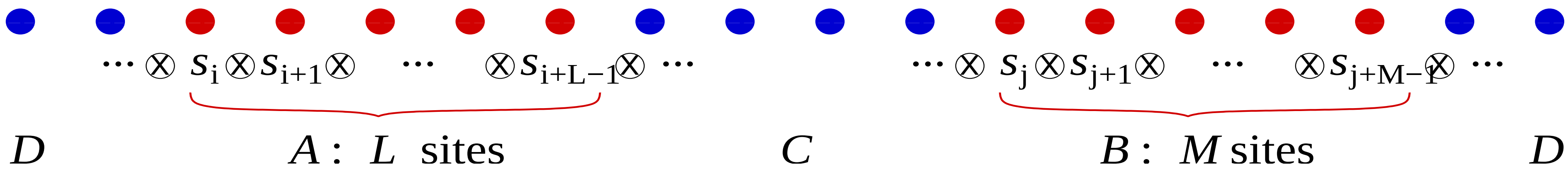
- For integrable models, an specific program exists (**form factor program**) that allows their exact (non-perturbative) computation.

- However the program needs to be modified to include twist fields correctly.

- Main challenge: analytic continuation from $n = 1, 2, \ldots$ to $n \in [1, \infty)$.

The entropy of disconnected regions

- Consider again a quantum spin chain which we now divide into four different regions (periodic boundary conditions)



- A problem of current interest is finding the entanglement entropy of the region $A \cup B$ with respect to the rest of the system (where A and B are now “disconnected”)

$$S_{A\cup B} = -\mathrm{Tr}_{A\cup B}(\rho_{A\cup B} \ln(\rho_{A\cup B}))$$

with

$$\rho_{A\cup B} = \mathrm{Tr}_{C\cup D}(|gs\rangle\langle gs|)$$

and $|gs\rangle$ is again the ground state of the chain (a pure state).

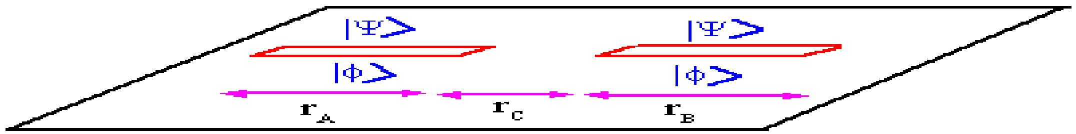
The entropy of disconnected regions and the replica trick

- We can use the “replica trick” for evaluating the entanglement entropy as before:

$$S_{A\cup B} = -\mathrm{Tr}_{A\cup B}(\rho_{A\cup B}\log(\rho_{A\cup B})) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_{A\cup B}(\rho^n_{A\cup B})$$

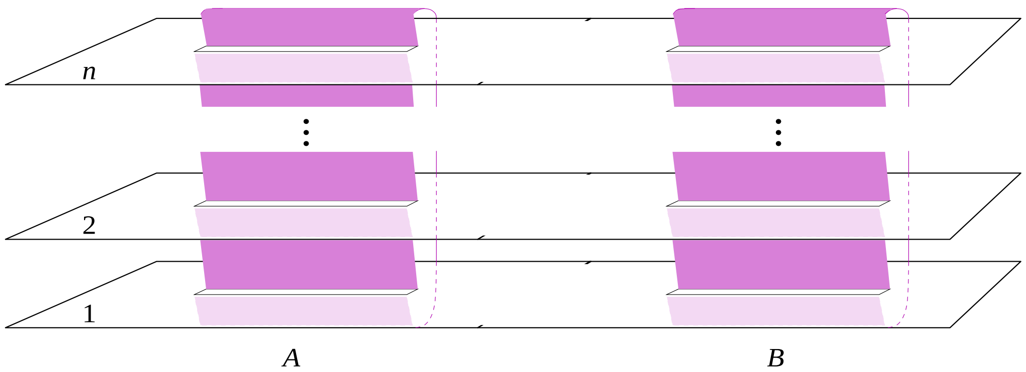
- For integer numbers n of replicas, in the scaling limit, this is again a partition function on a (more complicated) Riemann surface:

$${}_{A\cup B}\langle\phi|\rho_{A\cup B}|\psi\rangle_{A\cup B} \quad \sim$$



$$\mathrm{Tr}_{A\cup B}(\rho^n_{A\cup B}) \quad \sim \quad Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

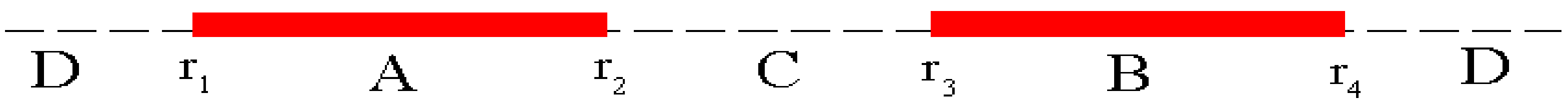
$$\mathcal{M}_n \quad :$$



The entropy of disconnected regions from twist fields

- Similarly to the previous case, the partition function of the n-sheeted Riemann surface can be expressed in terms of a correlation function of twist fields. In this case it is a four point function:

$$
\begin{aligned}
&\langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle_{\mathcal{L}^{(n)}} \\
&\propto \int_{C_{0,r}} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} d^2x\, \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x)\right] \\
&= Z_n = \mathrm{Tr}_{A\cup B}(\rho^n_{A\cup B})
\end{aligned}
$$



- From a computational point of view, it is much more challenging to compute a four-point function than a two-point function, even in the two-particle approximation.
- Some limiting cases, such as $r_{1,2} \ll r_{3,4}$ can be analyzed more easily.

Extensivity

- A property of the entropy of disconnected regions that has been recently investigated is its extensivity.
- Extensivity implies that the entropy of disconnected regions can be expressed in terms of the individual entropies of connected domains

$$S_{A\cup B} = S_A + S_B + S_C + S_D - \frac{1}{2}(S_{A\cup C} + S_{B\cup C} + S_{A\cup D} + S_{B\cup D})$$

- In our twist field picture it would imply a relationship of the type

$$\lim_{n\to 1} \frac{d}{dn} \langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle \sim \lim_{n\to 1} \frac{d}{dn} \Big[\langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\rangle + \langle \mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle$$
$$+ \langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_4)\rangle + \langle \mathcal{T}(r_2)\tilde{\mathcal{T}}(r_3)\rangle - \langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_3)\rangle - \langle \mathcal{T}(r_2)\tilde{\mathcal{T}}(r_4)\rangle \Big]$$

Some simple limit cases

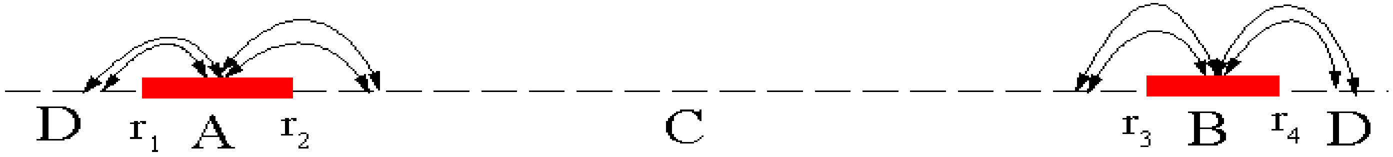
- When $r_1, r_2 \ll r_3, r_4$ the 4-point function factorizes as

$$\lim_{n\to 1}\frac{d}{dn}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle \to \lim_{n\to 1}\frac{d}{dn}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\rangle\langle\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle$$

$$= \underbrace{\lim_{n\to 1}\langle\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle}_{=1}\underbrace{\lim_{n\to 1}\frac{d}{dn}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\rangle}_{=-S_A}$$

$$+ \underbrace{\lim_{n\to 1}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\rangle}_{=1}\underbrace{\lim_{n\to 1}\frac{d}{dn}\langle\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle}_{=-S_B} = -S_A - S_B,$$

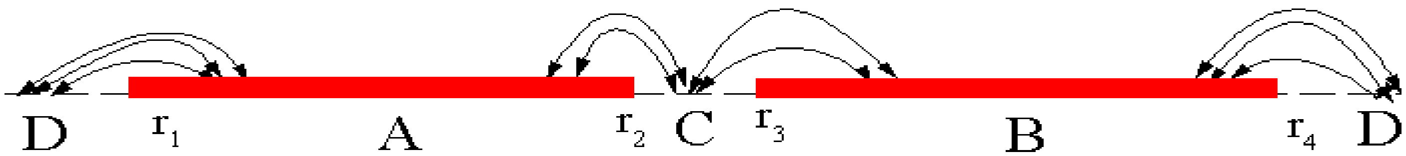
- Regions A and B are so far from each other (the size of region C tends to ∞ whilst the sizes of regions A and B are fixed and finite) that the bipartite entropy becomes simply the sum of the entanglement entropy between each of the regions and the rest of the system.



- When $r_1 \ll r_2$ and $r_3 \ll r_4$ the 4-point function factorizes as

$$\lim_{n\to1}\frac{d}{dn}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle \to \lim_{n\to1}\frac{d}{dn}\langle\mathcal{T}(r_1)\rangle\langle\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\rangle\langle\tilde{\mathcal{T}}(r_4)\rangle$$
$$= \lim_{n\to1}\langle\mathcal{T}\rangle^2\frac{d}{dn}\langle\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\rangle + \lim_{n\to1}\langle\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\rangle\frac{d}{dn}\langle\mathcal{T}\rangle^2$$
$$= -S_C + 2\lim_{n\to1}\frac{d\langle\mathcal{T}\rangle}{dn}$$

- In this case the size of regions A and B tends to ∞ while the size of region C is fixed and finite. Region C only sees the infinite regions A and B and its entanglement with them is S_C. To this we add the two "boundary" contributions from the entanglement of A and B with D.



- These two examples are compatible with extensivity of the entanglement entropy!

Non-extensivity

- In recent years numerical and analytical evidence of the non-extensivity of the entanglement entropy has been found. In fact, it appears that the entropy is only extensive for the massless free Fermion in 1+1 dimensions [Casini, Huerta 2004; Calabrese, Cardy 2004/2005, Casini, Fosco, Huerta 2005]
- [Caraglio, Gliozzi 2008] found analitical and numerical evidence (critical Ising model) of non-extensivity. [Casini, Huerta 2009] have also shown analytically the non-extensivity of the entropy for a massive free Fermion model and small mass. The entanglement entropy of disconnected regions has been shown to be non-extensive for generic CFTs in [Calabrese, Cardy, Toni 2009]. Numerical evidence for the free compactified Boson was also provided there.
- Numerical computations of the entanglement entropy of disconnected regions, showing its non-extensivity have been carried out for the critical Ising model [Alba, Tagliacozzo, Calabrese 2009] and for the XY integrable spin chain [Fagotti, Calabrese 2010].
- Preliminary results of our work appear to indicate that **extensivity is also violated when looking at higher order corrections** to the entanglement entropy, when the sizes of all regions are large and of the same order of magnitude.

Conclusions

- The main result of our work has been the derivation of the first **correction to saturation** of the entanglement entropy in any IQFT and the computation of all corrections for the Ising model (with and without boundaries).
- The key ingredients for this are the introduction of branch point twist fields in terms of whose two-point function the entropy can be evaluated. The form factor program has been generalised to accommodate **branch point twist fields**.
- In this talk I have tried to indicate how our approach could be pursued further to investigate the **entanglement entropy of disconnected regions** (two for a start).
- The next-to-leading order correction to the entanglement entropy of two disconnected regions involves multiple integrals over combinations of two- and four-particle form factors of the twist fields whose **analytic continuation** in n is extremely involved.
- Preliminary results suggest that some higher order correction to the entanglement entropy of two disconnected regions are **not compatible with extensivity**.