Dynamical Systems Coursework 3

To be handed in by 4 p.m. on Monday 28th March 2011

1. [**40 points**]Consider the following system of linear differential equations:

$$\frac{dx_1}{dt} = \dot{x}_1 = 2x_1 - \mu x_2 + 1, \qquad \frac{dx_2}{dt} = \dot{x}_2 = x_1 + 4x_2,$$

where μ is a constant.

(a) Write the equations in the standard form $\dot{\underline{x}} = A\underline{x} + \underline{b}$. **[2]**

(b) The system has a single fixed point. Find the fixed point. For which values of μ is it a simple fixed point? Justify your answer. **[4]**

(c) Compute the eigenvalues and eigenvectors of the matrix A for a particular value of μ of your choice (choose a value for which the fixed point is simple). Classify the fixed point. **[8]**

(d) For the same value of μ chosen in the previous section, define the vector $\underline{z}$ in terms of which the equation can be re-written in the form $\dot{\underline{z}} = A\underline{z}$. **[2]**

(e) For the same value of μ as before, write down the Jordan normal form J of the matrix A and the matrix P which enters the relation $J = P^{-1}AP$. Check $J = P^{-1}AP$ explicitly. Write down the relationship between the vector $\underline{z}$ above and the vector $\underline{y}$ in the canonical equation $\dot{\underline{y}} = J\underline{y}$. **[8]**

(f) For the same value of μ chosen in the previous section, write down the general solution for the vectors $\underline{x}$, $\underline{y}$ and $\underline{z}$. **[8]**

(g) For the same value of μ as before, sketch the phase diagrams of the system in the $x_1 - x_2$, $y_1 - y_2$ and $z_1 - z_2$ planes, clearly indicating their main features. **[8]**

2. [**30 points**]Consider the following second order nonlinear differential equation

$$\frac{d^2x}{dt^2} = \ddot{x} = \sin x - \cos x.$$

(a) Find a suitable change of variables that allows you to rewrite the equation above as a pair of first order differential equations for two variables x_1, x_2. **[2]**

(b) Find the fixed points of the system. **[4]**

(c) Find the general form of the Jacobian matrix of the system and evaluate it only at the fixed points with $-\frac{\pi}{2} < x_1 < \frac{\pi}{2}$. **[6]**

(d) Classify the fixed points in (c) according to their linearisation. **[6]**

(e) Solve the linearised equations for the same fixed points considered in (c) and (d). **[8]**

(f) Explain whether or not you think that the phase space diagram near the fixed points will look similar to the the phase space diagram of the linearised version of the system near the fixed points. **[4]**

3. [**30 points**]Consider the following system of first order nonlinear differential equations:

$$\frac{dx_1}{dt} = \dot{x}_1 = 1 - x_1 - x_2, \qquad \frac{dx_2}{dt} = \dot{x}_2 = x_1(x_2^2 - 1)(1 - x_1 - x_2).$$

(a) Show that all the fixed points of the system fall on a line. **[3]**

(b) Calculate the Jacobian matrix for a general point (x_1, x_2). **[5]**

(c) Use the result of (b) to determine whether or not the fixed points of section (a) are simple. **[5]**

(d) By dividing the two equations obtain a new equation of the form $\frac{dx_2}{dx_1} = X(x_1, x_2)$. Find the particular solution which satisfies the initial conditions $x_2 = 2$ if $x_1 = 0$. **[8]**

(e) Sketch the particular solution found in section (d) in the $x_1 - x_2$ plane. Indicate the values of x_1 for which the solution is well defined. Explain the behaviour of the function obtained as $x_1 \to \pm\infty$. Add arrows to your plot, as you would do for a phase space diagram. Does the direction of the arrows change when the trajectory above meets the line of fixed points obtained in (a)? Why? **[9]**