Dynamical Systems Coursework 3: Solutions and Feedback

1. (a)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & -\mu \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

[2]

(b) The fixed point is the solution of the equation $A\underline{a} + \underline{b} = 0$, that is $\underline{a} = -A^{-1}\underline{b}$. The fixed point is simple if $\det(A) = 8 + \mu \neq 0$, that is for all values of $\mu \neq -8$. In order to find the fixed point we need to evaluate the inverse of A:

$$A^{-1} = \frac{1}{8+\mu} \begin{pmatrix} 4 & \mu \\ -1 & 2 \end{pmatrix},$$

and multiply by $\underline{b}$,

$$\underline{a} = -\frac{1}{8+\mu} \begin{pmatrix} 4 & \mu \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{8+\mu} \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

[4]

(c) Now we have to choose a value of $\mu \neq -8$. **Here people took different values (basically everybody had either $\mu = 1$ or $\mu = 0$ with a few people taking other values). I will be doing the case $\mu = 0$ in detail. People who made a different choice and have questions can ask them in the revision lecture**.

For $\mu = 0$, the eigenvalues of A are obtained by solving

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) = 0,$$

which gives solutions $\lambda_1 = 4$ and $\lambda_2 = 2$. The eigenvalues are real, positive and different from each other, which means that the fixed point is an unstable node. **[2]**

We now just have to compute the eigenvectors. For $\underline{E}_1$ we solve,

$$\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix},$$

which gives equations $2a = 4a$ and $a + 4b = 4b$. This means that $a = 0$ and b can be taken to be any value (except 0). If we take $b = 1$, we find

$$\underline{E}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

[3]

For the other eigenvector we have to solve

$$\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix},$$

which gives equations $2a = 2a$ and $a + 4b = 2b$. If we take $a = 2$, then $b = -1$, which gives the second eigenvector

$$\underline{E}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

[3]

(d) The vector is just $\underline{z} = \underline{x} - \underline{a}$, with $\underline{a}$ the fixed point computed in (b). **Most people wrote more in the section, but really what I was asking was just this!** **[2]**

(e) The Jordan normal form is

$$J = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix},$$

[2]

and the matrix P is

$$P = (\underline{E}_1, \underline{E}_2) = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}.$$

[2]

This can be checked by computing

$$P^{-1}AP = -\frac{1}{2}\begin{pmatrix} -1 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -1 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ 4 & -2 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -8 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} = J.$$

[2]

The relationship between $\underline{y}$ and $\underline{z}$ is

$$\underline{z} = P\underline{y}.$$

[2]

Surprisingly, lots of people got this wrong here, although they used the right relation in the next section when writing the solutions. Many people wrote that $\underline{x} = P\underline{y}$. This is true when the fixed point is at the origin. When the fixed point is not at the origin the equation is not anymore $\dot{\underline{x}} = A\underline{x}$. It changes to $\dot{\underline{x}} = A\underline{x} + \underline{b}$. In order to make this equation look again like $\dot{\underline{x}} = A\underline{x}$ we introduce the vector $\underline{z}$ in terms of which $\dot{\underline{x}} = A\underline{x} + \underline{b}$ becomes $\dot{\underline{z}} = A\underline{z}$. Now we can use the relation $J = P^{-1}AP$ to transform this equation into $\dot{\underline{y}} = J\underline{y}$ if we take $\underline{z} = P\underline{y}$.

(f) The general solution for this kind of fixed point is always

$$\underline{y} = C_1 e^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

[3]

For $\underline{z}$ the solution is just P times the solution above, that is

$$\underline{z} = C_1 e^{4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

[3]

The solution for $\underline{x}$ is just $\underline{z}$ plus the fixed point, that is

$$\underline{x} = -\frac{1}{8}\begin{pmatrix} 4 \\ -1 \end{pmatrix} + C_1 e^{4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

[2]

(g) See figures in the figures file. Note that the only difference between the picture in the $z_1 - z_2$ coordinates and the $x_1 - x_2$ coordinates is position of the fixed point $\underline{a} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{8} \end{pmatrix}$. Otherwise the diagrams are identical. I awarded 2 points for the $y_1 - y_2$ diagram and 3 points for each of the other two. **[8]**

2. (a) The change of variables is as usual $x = x_1$ and $\dot{x} = x_2$. This implies that

$$\dot{x}_1 = x_2 \qquad \text{and} \qquad \dot{x}_2 = \sin x_1 - \cos x_1.$$

[2]

(b) The fixed points are the values of (x_1, x_2) for which the r.h.s. of both equations is zero. For the first equation there is just one solution $x_2 = 0$. When substituting into the 2nd equation we find that for $x_2 = 0$ we need $\sin x_1 = \cos x_1$, which corresponds to $x_1 = \frac{\pi}{4} + n\pi$ with $n = 0, \pm 1, \pm 2 \ldots$. In summary, there are infinitely many fixed points of the form

$$\left(\frac{\pi}{4} + n\pi, 0\right) \quad \text{with} \quad n = 0, \pm 1, \pm 2 \ldots$$

[4]

(c) The Jacobian matrix is

$$A_{(x_1,x_2)} = \begin{pmatrix} 0 & 1 \\ \cos x_1 + \sin x_1 & 0 \end{pmatrix}_{(x_1,x_2)}.$$

[2]

The fixed point that has $-\frac{\pi}{2} < x_1 < \frac{\pi}{2}$ is just $(\frac{\pi}{4}, 0)$. The Jacobian matrix at this point is

$$A_{(\frac{\pi}{4},0)} = \begin{pmatrix} 0 & 1 \\ \sqrt{2} & 0 \end{pmatrix}_{(\frac{\pi}{4},0)}$$

[4]

(d) In order to classify the fixed point we have to compute the eigenvalues of the matrix above. The eigenvalue equation is

$$\lambda^2 - \sqrt{2} = 0,$$

which has solutions $\lambda_1 = 2^{1/4}$ and $\lambda_2 = -2^{1/4}$. Therefore the eigenvalues are real with opposite signs. The fixed point is a saddle. **[6]**

(e) In order to solve the equations we need first to find the eigenvectors.

$$\begin{pmatrix} 0 & 1 \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2^{1/4} \begin{pmatrix} a \\ b \end{pmatrix},$$

which gives $b = 2^{1/4}a$ and $\sqrt{2}a = 2^{1/4}b$. As before, it is easy to see that both equations are in fact equivalent. One possible choice of eigenvector is to take $a = 1$ and $b = 2^{1/4}$ so that

$$\underline{E}_1 = \begin{pmatrix} 1 \\ 2^{1/4} \end{pmatrix}.$$

Similarly, the second eigenvector can be found to be

$$\underline{E}_2 = \begin{pmatrix} 1 \\ -2^{1/4} \end{pmatrix}.$$

This allows us to write down the linearized solution about this fixed point as

$$\underline{x} = \begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix} + C_1 e^{2^{1/4}t}\underline{E}_1 + C_2 e^{-2^{1/4}t}\underline{E}_2.$$

[8]

(f) The answer is that one would expect the non-linear system to be well approximated by the linearised solutions. This is because of the linearisation theorem that we saw in the class, namely that in two-dimensions the linear approximation is always good near a fixed point, except if the fixed point is a centre **Some people here confused the fixed point being a centre with the fixed point being at the origin, which are entirely different things**. [4]

3. (a) The fixed points are the solutions of

$$X_1(x_1, x_2) = 1 - x_1 - x_2 \quad \text{and} \quad X_2(x_1, x_2) = x_1(x_2^2 - 1)(1 - x_1 - x_2) = 0,$$

clearly one solution to this is $x_1 + x_2 = 1$ which is the equation of a line in the $x_1 - x_2$ plane. The second equation is also solved by $x_1 = 0$ and $x_2 = \pm 1$, but this does not solve the first equation, so it does not correspond to any extra fixed points. **Given that the question was telling you the answer already, it is surprising how many people got this wrong. To have a fixed point you need both equations $\dot{x}_1 = \dot{x}_2 = 0$ to be fulfilled at the same time. This happens for example if $x_1 = 0$ and $x_2 = 1$, but then this point is in the line $x_2 = 1 - x_1$ and therefore you don't need to consider it separately. If you do things carefully you will see that any point you find has to be on the line $x_2 = 1 - x_1$ and also that every point on the line is a fixed point.** [3]

(b) The Jacobian matrix is

$$A_{(x_1,x_2)} = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix}_{(x_1,x_2)} = \begin{pmatrix} -1 & -1 \\ (x_2^2-1)(1-2x_1-x_2) & 2x_1x_2(1-x_1-x_2)-x_1(x_2^2-1) \end{pmatrix}_{(x_1,x_2)}$$

[5]

(c) If we insert the condition $x_1 + x_2 = 1$ into the Jacobian matrix above we find

$$A_{(x_1,2-x_1)} = \begin{pmatrix} -1 & -1 \\ -(x_2^2-1)(1-x_2) & -(1-x_2)(x_2^2-1) \end{pmatrix}_{(x_1,2-x_1)}.$$

The determinant of this matrix is clearly zero, therefore the fixed points on the line are not simple. **[5]**

(d) The new equation is

$$\frac{dx_2}{dx_1} = (x_2^2-1)x_1,$$

this can be solved using separation of variables

$$\frac{dx_2}{x_2^2-1} = x_1 dx_1 \quad \Rightarrow \quad \int \frac{dx_2}{x_2^2-1} = \frac{x_1^2}{2} + C.$$

Integration in x_2 gives

$$\int \frac{dx_2}{x_2^2-1} = \frac{1}{2}\int \frac{dx_2}{x_2-1} - \frac{1}{2}\int \frac{dx_2}{x_2+1} = \log\sqrt{\frac{x_2-1}{x_2+1}}.$$

[2]

Therefore we find

$$\log\sqrt{\frac{x_2-1}{x_2+1}} = \frac{x_1^2}{2} + C \quad \Leftrightarrow \quad \frac{x_2-1}{x_2+1} = e^{x_1^2+2C} \quad \Leftrightarrow \quad x_2 = -\frac{e^{x_1^2+2C}+1}{e^{x_1^2+2C}-1}.$$

[2]

The constant C can be fixed using the initial condition $x_2 = 2$ for $x_1 = 0$. It gives

$$2 = -\frac{e^{2C}+1}{e^{2C}-1} \quad \Leftrightarrow \quad C = -\frac{1}{2}\log(3).$$

[2]

Therefore, the particular solution we were looking for is

$$x_2 = -\frac{e^{x_1^2}+3}{e^{x_1^2}-3}$$

[2]

(e) Since $x_2 = -\frac{e^{x_1^2}+3}{e^{x_1^2}-3}$ we see that when $x_1 \to \pm\infty$, $x_2 \to -1$. The function is also symmetric in x_1, that is it is invariant under the change $x_1 \to -x_1$. **[2]**
The function is well defined everywhere except when $e^{x_1^2} = 3$ which corresponds to $x_1^2 = \log(3)$. Therefore there are two values of x_1 for which this function is not well defined leading to the figure we will see later. **[2]**
These features allow for a fairly accurate sketch of the function, as can be seen in the figure (see pictures file). **[2]**
When the trajectory crosses the line $x_1 + x_2 = 1$ there should be a change in the direction of the

arrows. This is because the sign of $\dot{x}_1$ changes as we move from the region $x_1 + x_2 < 1$ to the region $x_1 + x_2 > 1$. The first region is on the l.h.s. of the line and in there $\dot{x}_1 > 0$, whereas in the second region on the r.h.s. of the line $\dot{x}_1 < 0$.

It is also possible to deduce this behaviour by looking at the equation for $\dot{x}_2$. However, in this case we must also consider the sign of the pre-factor $(x_2^2 - 1)x_1$ which makes things much more complicated. For $1 - x_1 - x_2 > 0$ (on the l.h.s. of the line), $\dot{x}_2 > 0$ if $x_1 > 0$ and $x_2^2 > 1$. This explains the ascending arrow on the r.h.s. of the picture. The arrow changes direction as it meets the line $1 - x_1 - x_2 = 0$ as on the r.h.s. of the line with $x_1 > 0$ and $x_2^2 > 1$ we have $\dot{x}_2 < 0$. One can similarly explain all other arrows in this way. **[3]**