Dynamical Systems Coursework 2

Optional Coursework (will not contribute to final grade)

This coursework is optional. This means that you do not need to attempt it or hand it in. If you attempt any of the questions and wish to have detailed feedback on your work from the lecturer you should hand your work in. Your work will not contribute towards your final mark for the module. Therefore, *only students with a genuine interest in having feedback from the lecturer and who have made a serious attempt to solve any of the questions* should hand in this coursework. There is no deadline for handing in this coursework.

1. Consider the following first order differential equation

$$\frac{dy}{dt} = \dot{y} = y(y-1),$$

(a) Find and classify the fixed points of the equation.

(b) Draw the phase space diagram associated to the equation.

(c) Find the solution to the equation for the following initial conditions (there will be a different solution for each case!):

- $y(0) = -1$,
- $y(0) = \frac{1}{2}$,
- $y(0) = 2$.

(d) For each solution, indicate the range of values of t for which it is defined. Indicate the behaviour of each solution for $t \to \pm\infty$.

(e) Sketch the solutions that you just obtained in the same diagram, drawing also the lines corresponding to the fixed points.

(f) Linearize the original equation about the fixed points. Solve the resulting linear equation for the initial condition $y(0) = \frac{1}{2}$.

2. Consider the following second order linear differential equation:

$$\ddot{y} - 2\dot{y} + 2y = 1,$$

(a) Use the methods that you learned in first year calculus to find the general solution to this equation.

(b) We saw in the lecture that an equation of this type can always be transformed into two first order linear differential equations by redefining $y = x_1$ and $\dot{y} = x_2$. Write down the new equations in terms of the variables x_1 and x_2.

(c) Write the equations of section (b) in the matrix form $\dot{\underline{x}} = A\underline{x} + \underline{b}$ which we have been using in the lecture.

(d) Find the fixed point of the system of equations, $\underline{a}$. Hence find $\underline{z}$ so that the original equation can be brought into the form $\dot{\underline{z}} = A\underline{z}$.

(e) Find the eigenvalues and eigenvectors of the matrix A.

(f) Find the Jordan Normal form of A. Construct the matrix P that relates the two matrices as $A = PJP^{-1}$.

(g) Find the general solution to the system of equations $\dot{\underline{y}} = J\underline{y}$.

(h) Using the solution found in part (g) find the solution for $\underline{z}$ and then for $\underline{x}$. Check that this solution is actually the same you found in section (a).

(i) Classify the fixed point at the origin and draw the phase space diagram both in the $y_1 - y_2$ and in the $x_1 - x_2$ planes.