Solutions to Dynamical Systems 2010 exam. Each question is worth 25 marks.

1. ***[Unseen]*** *Consider the following 1st order differential equation:*

$$\frac{dy}{dt} = X(y) = -y(y^2-4). \tag{1}$$

(a) *Find and classify all the fixed points of (1). Hence draw the phase diagram associated to (1).*
We just need to find the solutions to $X(y)=0$, which are $y=0,\pm 2$. In order to classify the fixed points we have to either look at the sign of the r.h.s. of (1) in the various regions between fixed points or use the criterium coming from linearisation which means looking at the sign of the derivative of the r.h.s. of (1) at each of the fixed points. Following this last criterium we find that,

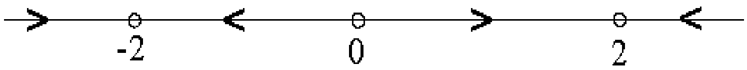
$$X'(y) = -(y^2-4) - y(2y) = -3y^2+4.$$

So we find that:

- $X'(0)=4>0$, therefore $y=0$ is a repellor,
- $X'(-2)=-8>0$, therefore $y=-2$ is an attractor,
- $X'(2)=-8<0$,therefore $y=2$ is also an attractor.

[3 marks]

Hence, the phase diagram is



[1 mark]

(b) *Solve (1) for each of the following initial conditions:*

- $y(0)=3$,
- $y(0)=-1$,

indicating the range of values of t for which each solution is defined
The equation can be solved by separating variables in the usual way:

$$\begin{aligned} \int_{t_0}^{t} dt &= t-t_0 = -\int_{y_0}^{y} \frac{dy}{y(y-2)(y+2)} = -\int_{y_0}^{y}\left(\frac{1}{8\,(-2+y)} - \frac{1}{4\,y} + \frac{1}{8\,(2+y)}\right) \\ &= -\frac{1}{8}\log\left|\frac{(y^2-4)y_0^2}{(y_0^2-4)y^2}\right|, \end{aligned}$$

with $y_0 = y(t_0)$ the initial condition. This can be solved to

$$y(t) = \pm\frac{2e^{4(t-t_0)}}{\sqrt{e^{8(t-t_0)}-1+\frac{4}{y_0^2}}}.$$

The $\pm$ sign will be $+$ for $y_0>0$ and $-$ for $y_0<0$ in order to satisfy the initial condition.

[5 marks]

Let us now solve for each initial condition:

- for $y(0)=3$ we have

$$y(t) = \frac{2e^{4t}}{\sqrt{e^{8t}-\frac{5}{9}}}.$$

This solution is well defined only when $e^{8t}-\frac{5}{9}>0$. This gives the condition

$$t > \frac{1}{8}\log\frac{5}{9} = -0.0734733.$$

Thus the range of values for which the solution is defined is $t \in (\frac{1}{8}\log\frac{5}{9}, \infty)$.

[4 marks]

- For $y(0) = -1$ the solution becomes,

$$y(t) = -\frac{2e^{4t}}{\sqrt{e^{8t}+3}},$$

where we had to introduce a minus sign to guarantee that the initial condition is fulfilled. In this case, the denominator is never zero and $e^{8t}+3$ can never become negative, so the solution is valid for all values of t. The interval of definition is $t \in (-\infty, \infty)$.

[4 marks]

(c) *Provide a sketch of your solutions against the variable t, clearly showing the behaviour of the various functions as $t \to \pm\infty$.*
In order to carry out the sketch it is important to study the behaviour of both solutions as $t \to \pm\infty$. The solution

$$y(t) = \frac{2e^{4t}}{\sqrt{e^{8t} - \frac{5}{9}}}, \quad y(0) = 3,$$

is clearly always positive (for those values of t for which it is defined) and as $t \to \infty$ it tends to the value 2, which is the fixed point we identified before. This was to be expected, as the point is an attractor. In this case it does not make sense to consider $t \to -\infty$ as the solution is only defined for $t \in (\frac{1}{8}\log\frac{5}{9}, \infty)$. Finally, since the denominator vanishes exactly at $t = \frac{1}{8}\log\frac{5}{9}$ the function should diverge as it gets closer to that value.

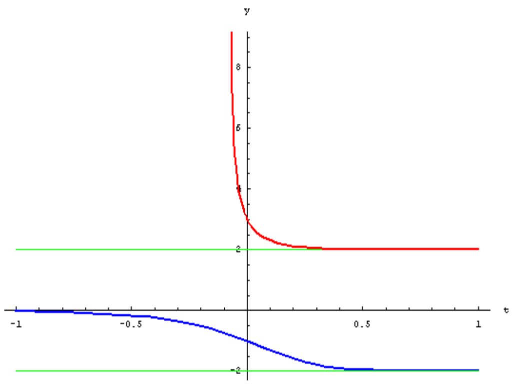


Figure 1: The red figure corresponds to the initial condition $y(0) = 3$. As we can see it diverges at a small negative value of t which was computed above. The blue graph is the solution with $y(0) = -1$. The green lines are the fixed points $y = \pm 2$.

Concerning the other solution

$$y(t) = -\frac{2e^{4t}}{\sqrt{e^{8t}+3}}, \quad y(0) = -1,$$

it is clearly a negative function for all values of t. As $t \to \infty$ it tends to the value -2 which is the fixed point identified before. We should have expected that, since -2 is an attractor. As $t \to -\infty$ the solution tends to the value 0, which is also compatible with our fixed point analysis, as zero was identified as a repellor. A plot of the two solutions is given above.

[5 marks]

(d) *Write down the linearized version of equation (1) about a generic fixed point $y = a$.*
At a fixed point $X(a) = 0$ and therefore a Taylor expansion of $X(y)$ to first order in y gives

$$\frac{dy}{dt} = X'(a)(y-a) = (-3a^2+4)(y-a).$$

[1 mark]

(e) *Use the result of (d) to find the solution of the linearized equation about the smallest of the fixed points of (1), if the initial condition is $y(0) = -3$.* We already found in (a) that $X'(-2) = -8$ therefore the linearized equation is

$$\frac{dz}{dt} = -8z,$$

for $z = y + 2$. The solution to this equation is $z = Ae^{-8t}$ where A is a constant which will be fixed by the initial condition. The final answer is:

$$y(t) = -2 - e^{-8t}.$$

[2 marks]

2. ***[Seen]*** *Consider the following system of linear differential equations:*

$$\frac{dx_1}{dt} = \dot{x}_1 = \alpha x_1 + \gamma x_2 + c_1, \qquad \frac{dx_2}{dt} = \dot{x}_2 = \beta x_1 + \alpha x_2 + c_2, \tag{2}$$

where $\alpha, \beta, \gamma, c_1, c_2$ are real constants.

(a) *Write the equations in the standard form $\dot{\underline{x}} = A\underline{x} + \underline{b}$. Write down the vector $\underline{a}$ which is a fixed point of (2).*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

[1 mark]

The fixed point is the solution to the vector equation

$$A\underline{x} + \underline{b} = 0,$$

that is

$$\underline{x} = -A^{-1}\underline{b} = -\frac{1}{\alpha^2 - \beta\gamma} \begin{pmatrix} \alpha & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} \frac{\alpha c_1 - \gamma c_2}{\alpha^2 - \beta\gamma} \\ \frac{-\beta c_1 + \alpha c_2}{\alpha^2 - \beta\gamma} \end{pmatrix}$$

[4 marks]

(b) *Compute the eigenvalues and eigenvectors of the matrix A.* The eigenvalues of A are obtained by finding the zeroes of the characteristic polynomial of A:

$$|A - \lambda I| = (\alpha - \lambda)^2 - \beta\gamma = (\alpha^2 - \beta\gamma) + \lambda^2 - 2\alpha\lambda = 0,$$

the solutions to this equation are

$$\lambda_1 = \frac{2\alpha + \sqrt{4\alpha^2 - 4(\alpha^2 - \beta\gamma)}}{2} = \alpha + \sqrt{\beta\gamma},$$

and

$$\lambda_2 = \frac{2\alpha - \sqrt{4\alpha^2 - 4(\alpha^2 - \beta\gamma)}}{2} = \alpha - \sqrt{\beta\gamma}.$$

[3 marks]

The eigenvectors corresponding to these eigenvalues are obtained by solving:

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (\alpha \pm \sqrt{\beta\gamma}) \begin{pmatrix} a \\ b \end{pmatrix}.$$

In components this gives two equations for each eigenvector:

$$a\alpha + b\gamma = (\alpha \pm \sqrt{\beta\gamma})a, \qquad a\beta + b\alpha = (\alpha \pm \sqrt{\beta\gamma})b,$$

simplifying

$$b\gamma = \pm\sqrt{\beta\gamma}a, \qquad a\beta = \pm\sqrt{\beta\gamma}b.$$

The two equations are not independent and setting $a = 1$ this gives $b = \pm\sqrt{\frac{\beta}{\gamma}}$. Therefore the eigenvectors would be:

$$\underline{E}_1 = \begin{pmatrix} 1 \\ \sqrt{\frac{\beta}{\gamma}} \end{pmatrix}, \qquad \underline{E}_2 = \begin{pmatrix} 1 \\ -\sqrt{\frac{\beta}{\gamma}} \end{pmatrix}$$

[5 marks]

(c) *Define the vector $\underline{z}$ in terms of which the equation can be re-written in the form $\dot{\underline{z}} = A\underline{z}$.*
We just need to change variables to

$$\underline{z} = \underline{x} - \underline{a},$$

where $\underline{a}$ is the fixed point obtained in section (a).

[1 mark]

(d) *Indicate the conditions (if any) that α, β and γ must satisfy so that the fixed point is:*

- *an improper node,*
- *a focus,*
- *a star node.*

An improper node arises when the eigenvalues of the matrix A are equal and real. In our case the eigenvalues can only be equal if

$$\sqrt{\beta\gamma} = 0,$$

which corresponds to $\beta = 0$ or $\gamma = 0$.

[1 mark]

A focus occurs when the eigenvalues are complex conjugated to each other. This will happen only if $\beta > 0$ and $\gamma < 0$ or viceversa.

[1 mark]

Finally, a star node occurs when the matrix A is proportional to the identity matrix. This will only happen if $\beta = \gamma = 0$.

[1 mark]

(e) *If $\alpha = 1$, $\beta = -2$ and $\gamma = 2$ classify the fixed point. Find the values of c_1 and c_2 so that the fixed point is $\underline{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.*
We found the fixed point in section (a)

$$\underline{a} = -\begin{pmatrix} \frac{\alpha c_1 - \gamma c_2}{\alpha^2 - \beta\gamma} \\ \frac{-\beta c_1 + \alpha c_2}{\alpha^2 - \beta\gamma} \end{pmatrix} = -\begin{pmatrix} \frac{c_1 - 2c_2}{5} \\ \frac{2c_1 + c_2}{5} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This is solved by $c_1 = -3$ and $c_2 = 1$.

[2 marks]

(f) *For $\alpha = 1$, $\beta = -2$ and $\gamma = 2$ show that the matrix A is already in its Jordan Normal form. For these values of α, β, γ and the values of c_1, c_2 obtained in (e) find the general solution of equations (2)*
For the values of α, β, γ given in (e), the eigenvalues of A are $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. The Jordan form is therefore:

$$J = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

which coincides with the matrix A itself.

[1 mark]

We will use the change of variables of section (c) so that the equation becomes of the form

$$\dot{\underline{z}} = A\underline{z},$$

where

$$\underline{z} = \underline{x} - \underline{a},$$

and

$$\underline{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

from section (e). The equation in $\underline{z}$ can be written in components as:

$$\dot{z}_1 = z_1 + 2z_2, \qquad \dot{z}_2 = -2z_1 + z_2,$$

and can be solved by using the standard change of variables

$$z_1 = r\cos\theta, \qquad z_2 = r\sin\theta.$$

This leads to the equations

$$\dot{r} = r, \qquad \Rightarrow \qquad r = Ae^t,$$

where A is a constant, and

$$\dot{\theta} = -2, \qquad \theta = -2t + B,$$

where B is an integration constant. Therefore, the general solutions are

$$z_1 = Ae^t\cos(-2t + B), \qquad z_2 = Ae^t\sin(-2t + B).$$

Therefore, for the original variables x_1, x_2 the general solutions are

$$x_1 = 1 + Ae^t\cos(-2t + B), \qquad x_2 = 1 + Ae^t\sin(-2t + B).$$

or, in vector form

$$\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + Ae^t\cos(-2t + B)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + Ae^t\sin(-2t + B)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

[5 marks]

3. *[Seen] Consider the following system of first order nonlinear differential equations:*

$$\dot{x}_1 = x_1(1 - x_1 - x_2), \qquad \dot{x}_2 = x_2(1 - x_1 - x_2). \tag{3}$$

(a) *Show that the fixed points of the system are either at the origin or on a line.*
In order to find the fixed points we have to solve the simultaneous equations:

$$x_1(1 - x_1 - x_2) = x_2(1 - x_1 - x_2) = 0,$$

which give solutions $(x_1, x_2) = (0, 0)$ and $x_1 = 1 - x_2$ which is indeed a line.

[2 marks]

(b) *Calculate the Jacobian matrix for a general point (x_1, x_2) and deduce that the fixed point at the origin is simple, whereas the fixed points on the line are not.*
The Jacobian matrix is obtained as usual by computing the derivatives

$$A_{(x_1,x_2)} = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 - 2x_1 - x_2 & -x_1 \\ -x_2 & 1 - 2x_2 - x_1 \end{pmatrix}$$

with $X_1(x_1, x_2) = x_1(1 - x_1 - x_2)$ and $X_2(x_1, x_2) = x_2(1 - x_1 - x_2)$.

[3 marks]

Evaluating this matrix at the origin we obtain

$$A_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is the identity matrix. The determinant of this matrix is obviously 1 and therefore non-vanishing. So the origin is a simple fixed point.

[2 marks]

For any point $(\alpha, 1-\alpha)$ lying on the line $x_1 = 1 - x_2$ we find that the Jacobian matrix is

$$A_{(\alpha,1-\alpha)} = \begin{pmatrix} -\alpha & -\alpha \\ -1+\alpha & -1+\alpha \end{pmatrix},$$

which has determinant $\det(A) = 0$, therefore the line of fixed points $x_1 = 1 - x_2$ contains only non-simple fixed points.

[3 marks]

(c) *By writing down the linearization about the fixed point at the origin classify its nature. Solve the linearized equation.*

We have already almost answered this question above when we computed the Jacobian matrix at the origin. The linearized equation about the origin is:

$$\underline{x} = A_{(0,0)}\underline{x},$$

which $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and since $A_{(0,0)}$ is the identity matrix, the fixed point at the origin is an <u>unstable star node</u>.

[2 marks]

The equation is very easy to solve. In components we have that

$$\dot{x}_1 = x_1, \qquad \dot{x}_2 = x_2,$$

which is solved by

$$x_1 = C_1 e^t, \qquad x_2 = C_2 e^t,$$

with C_1, C_2 arbitrary constants. Since $x_1/x_2 = C_1/C_2$, phase space trajectories near the fixed point will be straight lines going by the origin.

[2 marks]

(d) *By dividing the two equations in (3) obtain a new equation of the form $\frac{dx_1}{dx_2} = X(x_1, x_2)$. Find general solution to this equation. Explain why your solution is what one would expect given the nature of the fixed point at the origin.*

The resulting equation is

$$\frac{dx_1}{dx_2} = \frac{x_1(1-x_1-x_2)}{x_2(1-x_1-x_2)} = \frac{x_1}{x_2},$$

which only is well defined when $x_1 \neq 1 - x_2$ and $x_2 \neq 0$. The equation can be solved by separation of variables,

$$\int \frac{dx_1}{x_1} = \int \frac{dx_2}{x_2},$$

which gives

$$\ln x_1 = \ln x_2 + C,$$

or

$$x_1 = e^C x_2,$$

where C is an integration constant. Therefore, in general phase space trajectories are straight lines going by the origin.

[3 marks]

This is what one expects from the fact that the origin is an unstable star node which is precisely characterized by trajectories which are straight lines with various slopes, depending on the initial condition.

[2 marks]

(e) *Use the results of (c) and (d) to sketch the phase space diagram associated to (3). Pay special attention to the behaviour of phase space trajectories near the line of fixed points found in (a). Does the direction of the trajectories change at this line? Why?*

The phase space diagram is given in the figure below.

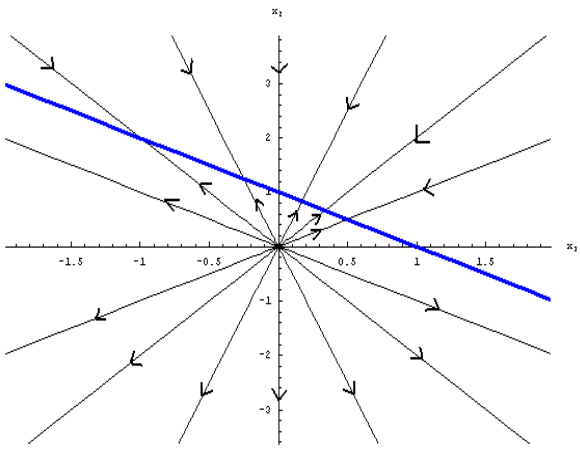


Figure 2: The phase space diagram of equation (3). The blue line is the line of fixed points $x_2 = 1 - x_1$.

[3 marks]

The phase space trajectories change direction exactly at the line $x_1 = 1 - x_2$ this is because the sign of the derivatives $\dot{x}_1$ and $\dot{x}_2$ changes at this line. Below the line, that is for $x_1 < 1 - x_2$ we have that $(1 - x_1 - x_2) > 0$, therefore

$$\dot{x}_1 = x_1(1 - x_1 - x_2) > 0, \qquad \dot{x}_2 = x_2(1 - x_1 - x_2) > 0,$$

if both $x_1, x_2 > 0$. This is compatible with trajectories moving away from the origin, in the direction of increasing x_1 and x_2. However for $x_1 > 1 - x_2$, that is above the line of fixed points we have exactly the opposite behaviour, that is

$$\dot{x}_1 = x_1(1 - x_1 - x_2) < 0, \qquad \dot{x}_2 = x_2(1 - x_1 - x_2) < 0,$$

if both $x_1, x_2 > 0$. Therefore, here the trajectories have to move towards the origin at time grows, that is in the direction of decreasing values of x_1 and x_1.

[3 marks]

4. *[Unseen] Consider the following second order linear differential equation:*

$$\ddot{x} - 4\dot{x} - 5x = 0, \tag{4}$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ *and* $\dot{x} = \frac{dx}{dt}$.

(a) *Consider now the following two first order linear differential equations,*

$$\dot{x}_1 = ax_1 + bx_2 + c, \qquad \dot{x}_2 = dx_1 + ex_2 + f, \tag{5}$$

where $x_1 = x$ above. Find the values of the constants a, b, c, d, e and f so that the system (5) is equivalent to (4).
By defining $x_1 = x$ and $x_2 = \dot{x} = \dot{x}_1$, we can rewrite (4) as:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 4x_2 + 5x_1,$$

so $a = c = f = 0$ and $b = 1$, $d = 5$, $e = 4$.

[2 marks]

(b) *Using the result of (a) write equation (4) in the standard matrix form $\underline{\dot{x}} = A\underline{x} + \underline{b}$.*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

with $\underline{b} = 0$ in this case.

[1 mark]

(c) *Find the eigenvalues and eigenvectors of the matrix A. Hence construct the Jordan normal form of A and the matrix P which relates A to its Jordan form J as $A = PJP^{-1}$. The eigenvalues of A are obtained by finding the zeroes of the characteristic polynomial of A:*

$$|A - \lambda I| = -\lambda(4 - \lambda) - 5 = 0 \quad \Rightarrow \quad \lambda_1 = -1, \qquad \lambda_2 = 5.$$

[2 marks]

The eigenvectors corresponding to these eigenvalues are obtained by solving:

$$\begin{pmatrix} 0 & 1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (-1) \begin{pmatrix} a \\ b \end{pmatrix}.$$

In components this gives two equations:

$$b = -a, \qquad 5a + 4b = -b,$$

which are not independent. Taking $a = 1$ we find that the eigenvector associated to $\lambda_1 = -1$ is

$$\underline{E}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

[2 marks]

and for $\lambda_2 = 5$ we obtain the equation

$$\begin{pmatrix} 0 & 1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 5 \begin{pmatrix} a \\ b \end{pmatrix}.$$

In components this gives two equations:

$$b = 5a, \qquad 5a + 4b = 5b,$$

which are not independent. Taking $a = 1$ we find that the second eigenvector is

$$\underline{E}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

[2 marks]

The Jordan normal form of A is just the diagonal matrix

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix},$$

and the matrix P is built just by setting its columns to the eigenvectors above:

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 5 \end{pmatrix}.$$

[2 marks]

(d) *Find the fixed point of the system of equations and classify its nature.*
In this case the fixed point is just at the origin.

[1 mark]

Because the eigenvalues are real and one eigenvalue is negative and the other positive this would be a saddle.

[3 marks]

(e) *Find the general solution of the system of equations $\dot{\underline{y}} = J\underline{y}$. Hence find the general solution for $\underline{x}$.*
In this case the canonical system of equations is simply:

$$\dot{y}_1 = -y_1, \qquad \dot{y}_2 = 5y_2,$$

which is solved by $y_1 = C_1 e^{-t}$ and $y_2 = C_2 e^{5t}$, where C_1 and C_2 are arbitrary constants.

[3 marks]

Now we need to take the solutions from part (e) and multiply them by the matrix P to obtain $\underline{x}$:

$$\underline{x} = P\underline{y} = \begin{pmatrix} 1 & 1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ -y_1 + 5y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{-t} + C_2 e^{5t} \\ -C_1 e^{-t} + 5C_2 e^{5t} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This can be rewritten as

$$\underline{x} = C_1 e^{-t} \underline{E}_1 + C_2 e^{5t} \underline{E}_2.$$

[3 marks]

(f) *Sketch the phase diagram both in the $y_1 - y_2$ plane and in the $x_1 - x_2$ plane.*

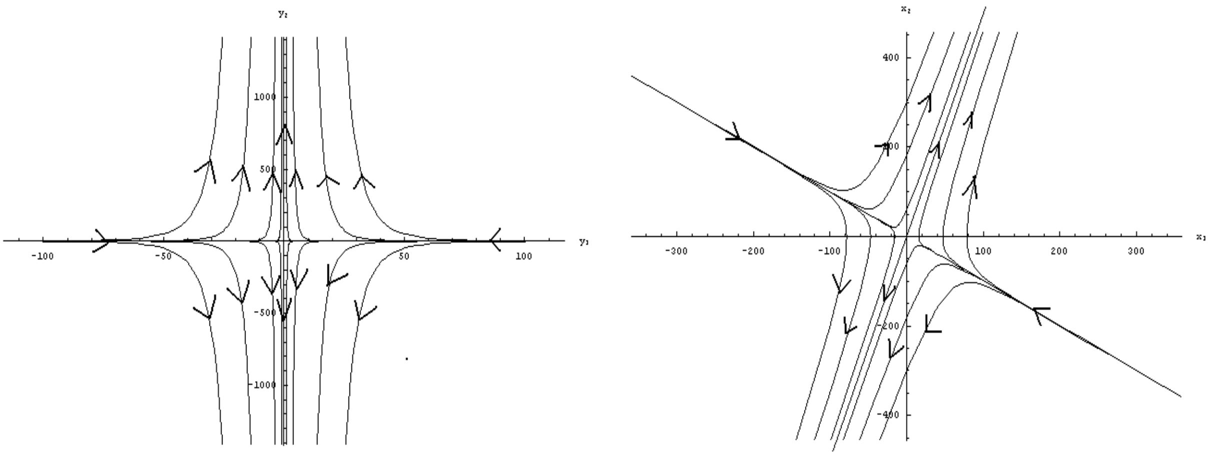


Figure 3: The $y_1 - y_2$ diagram is a typical saddle. As expected, the $x_1 - x_2$ diagram would be a rotated and slightly stretched version of the first diagram, where the symmetry axes are now the directions determined by the eigenvectors $\underline{E}_1$, $\underline{E}_2$.

[4 marks]

5. *[Unseen] Consider the following first order nonlinear differential equations*

$$\dot{x}_1 = (1 - \frac{x_1}{2} - x_2)x_1, \qquad \dot{x}_2 = -(1 + \frac{x_2}{2} - x_1)x_2, \tag{6}$$

which can be interpreted as modelling the populations of two competing species.

(a) *Find the four fixed points of the system.*
To find the fixed points we have to solve the simultaneous equations:

$$(1-\frac{x_1}{2}-x_2)x_1=0, \qquad (1+\frac{x_2}{2}-x_1)x_2=0.$$

An obvious solution is $(x_1,x_2)=(0,0)$. For $x_1=0$ we can also have $x_2=-2$ and for $x_2=0$ we can also have $x_1=2$. Finally, if both x_1 and x_2 are not zero there is also the solution $(x_1,x_2)=(\frac{6}{5},\frac{2}{5})$ is also a fixed point. Therefore, we have the four points:

$$(0,0), \quad (2,0), \quad (0,-2), \quad (\frac{6}{5},\frac{2}{5}).$$

[2 marks]

(b) *Find the general form of the Jacobian matrix of the system and evaluate it at each fixed point. Classify the fixed points according to their linearization.*
The Jacobian matrix is obtained by computing the derivatives of the r.h.s. of the equations (6) w.r.t. x_1 and x_2. Defining $X_1(x_1,x_2)=(1-x_1/2-x_2)x_1$ and $X_2(x_1,x_2)=-(1+x_2/2-x_1)x_2$ we obtain,

$$A=\begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix}=\begin{pmatrix} 1-x_1-x_2 & -x_1 \\ x_2 & -1-x_2+x_1 \end{pmatrix}$$

[2 marks]

At the various fixed points it becomes:

$$A_{(0,0)}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is diagonal with real eigenvalues of different signs. This means that the origin is a <u>saddle</u>.

[1 mark]

$$A_{(2,0)}=\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}.$$

In this case we need to find the eigenvalues of $A_{(2,0)}$ in order to classify the fixed point. They are the solution to the equation

$$(-1-\lambda)(1-\lambda)=0,$$

that is $\lambda=1$ and $\lambda=-1$. Therefore, as before this would also be a <u>saddle</u>.

[2 marks]

$$A_{(0,-2)}=\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}.$$

The eigenvalues of $A_{(0,-2)}$ are the solutions to the equation

$$(3-\lambda)(1-\lambda)=0,$$

that is $\lambda=1$ and $\lambda=3$. Therefore both eigenvalues are real and positive which corresponds to an <u>unstable node</u>.

[2 marks]

Finally,

$$A_{(6/5,2/5)}=\begin{pmatrix} -\frac{3}{5} & -\frac{6}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix}.$$

The eigenvalues of $A_{(6/5,2/5)}$ are the solutions to the equation

$$(-\frac{3}{5}-\lambda)(-\frac{1}{5}-\lambda)+\frac{12}{25}=\lambda^2+\frac{4\lambda}{5}+\frac{3}{5}=0,$$

that is $\lambda=-\frac{2}{5}\pm i\frac{\sqrt{11}}{5}$. These are complex conjugated eigenvalues with negative real part which would describe a <u>stable focus</u>.

[2 marks]

(c) *For only those fixed points lying in the region $-1.5 \leq x_1 \leq 1.5$ and $-1.5 \leq x_2 \leq 1.5$ solve the linearized equations.*

The only fixed points lying on the indicated region are $(0,0)$ and $(6/5, 2/5) = (1.2, 0.4)$. The first fixed point is a saddle and the linearized equation is already in the canonical form. Therefore, the general solution to this linearized equation is:

$$\underline{x} = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_1 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where C_1, C_2 are arbitrary constants.

[1 mark]

The fixed point $(6/5, 2/5) = (1.2, 0.4)$ is a stable focus. In order to solve the linearized equation about the fixed point we need to carry out the standard transformations. First we change variables to $\underline{z} = \underline{x} - \underline{a}$, where $\underline{a}$ is the fixed point so that we bring the fixed point to the origin. Second we rewrite the resulting linearized equation $\dot{\underline{z}} = A\underline{z}$ in the canonical form and solve this equation:

$$\dot{\underline{y}} = J\underline{y},$$

where J is the Jordan form of A and $\underline{z} = P\underline{y}$, where P is a nonsingular matrix that relates A and J as

$$A = PJP^{-1}.$$

[1 mark]

In order to construct the matrix P we need the eigenvectors of A. The eigenvalues $\lambda = -\frac{2}{5} \pm i\frac{\sqrt{11}}{5}$ mean that the Jordan form of the matrix A is

$$J = \begin{pmatrix} -\frac{2}{5} & \frac{\sqrt{11}}{5} \\ -\frac{11}{5} & -\frac{2}{5} \end{pmatrix}.$$

The eigenvectors of A are obtained by solving the equations

$$\begin{pmatrix} -\frac{3}{5} & -\frac{6}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \left(-\frac{2}{5} \pm i\frac{\sqrt{11}}{5}\right) \begin{pmatrix} a \\ b \end{pmatrix},$$

which can be written as,

$$-\frac{3a}{5} - \frac{6b}{5} = \left(-\frac{2}{5} \pm i\frac{\sqrt{11}}{5}\right) a, \qquad \frac{2a}{5} - \frac{b}{5} = \left(-\frac{2}{5} \pm i\frac{\sqrt{11}}{5}\right) b.$$

As usual, the two equations for each eigenvalue are not independent. Taking $a = 1$ we obtain $b = -\left(\frac{1}{6} \pm i\frac{\sqrt{11}}{6}\right)$, therefore the eigenvectors and eigenvalues are

$$\underline{E}_1 = \begin{pmatrix} 1 \\ -\frac{1}{6} - i\frac{\sqrt{11}}{6} \end{pmatrix}, \qquad \lambda_1 = -\frac{2}{5} + i\frac{\sqrt{11}}{5},$$

and

$$\underline{E}_2 = \begin{pmatrix} 1 \\ -\frac{1}{6} + i\frac{\sqrt{11}}{6} \end{pmatrix}, \qquad \lambda_2 = -\frac{2}{5} - i\frac{\sqrt{11}}{5}.$$

[3 marks]

The matrix P that relates A to its Jacobian form is obtained by taking the real and imaginary parts of the first eigenvector, and setting them as columns of J,

$$P = \begin{pmatrix} 1 & 0 \\ -\frac{1}{6} & -\frac{\sqrt{11}}{6} \end{pmatrix},$$

and we can check that indeed $A = PJP^{-1}$.

[1 mark]

The equation $\dot{\underline{y}} = J\underline{y}$ can be solved in the usual way, for J of the form above (see section (g) of question 2). The solution takes the form

$$\underline{y} = C_1 e^{-\frac{2t}{5}} \cos(\frac{\sqrt{11}}{5}t + C_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_1 e^{-\frac{2t}{5}} \sin(\frac{\sqrt{11}}{5}t + C_2) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where C_1, C_2 are constants.

[2 marks]

In order to get the solutions for the variable $\underline{z}$ we need to multiply the vector above by the matrix P and finally, to get $\underline{x}$ we have to add the fixed point

$$\underline{x} = \underline{a} + P\underline{y} = \begin{pmatrix} \frac{6}{5} \\ \frac{2}{5} \end{pmatrix} + C_1 e^{-\frac{2t}{5}} \cos(\frac{\sqrt{11}}{5}t + C_2) P \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_1 e^{-\frac{2t}{5}} \sin(\frac{\sqrt{11}}{5}t + C_2) P \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{5} \\ \frac{2}{5} \end{pmatrix} + C_1 e^{-\frac{2t}{5}} \cos(\frac{\sqrt{11}}{5}t + C_2) \begin{pmatrix} 1 \\ -\frac{1}{6} \end{pmatrix} + C_1 e^{-\frac{2t}{5}} \sin(\frac{\sqrt{11}}{5}t + C_2) \begin{pmatrix} 0 \\ -\frac{\sqrt{11}}{6} \end{pmatrix}.$$

In components:

$$x_1 = \frac{6}{5} + C_1 e^{-\frac{2t}{5}} \cos(\frac{\sqrt{11}}{5}t + C_2)$$

$$x_2 = \frac{2}{5} - \frac{C_1}{6} e^{-\frac{2t}{5}} \cos(\frac{\sqrt{11}}{5}t + C_2) - \frac{\sqrt{11}C_1}{6} e^{-\frac{2t}{5}} \sin(\frac{\sqrt{11}}{5}t + C_2).$$

[3 marks]

(d) *In the same region of the $x_1 - x_2$ plane considered in section (c) and using the solutions to the linearized equations, sketch the phase diagram associated to the nonlinear system of equations. You may employ the vector field diagram below as a guiding tool in order to produce your sketch.*

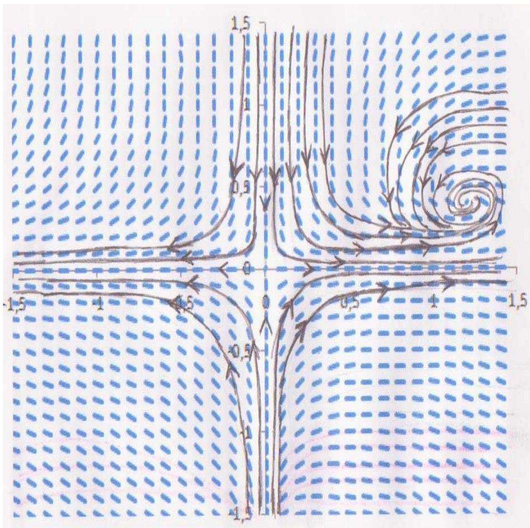


Figure 4: The sketch here is not entirely accurate, as I have ignored the presence of the other fixed points at (0,-2) and (2,0). Specially (0,-2) would affect the shape of the trajectories on the lower right hand corner of the picture, as one can see from the vector field diagram. The students were not expected to notice this.

[3 marks]