# Dynamical Systems Year 2

# COURSE HANDBOOK 2010 - 2011

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# Chapter 1

# Introduction

# 1.1 Introduction

The course is split into two distinct parts, initially looking at first order or one dimensional systems and then moving on to two and possibly higher dimensional systems. In the introduction we give examples where we introduce first order systems in their own right as well as use the solution of a first order system to investigate a second order one.

The first natural question to ask is what is a Dynamical System.

We will start by defining a Nonlinear System. A Nonlinear System is a set of (one or more) nonlinear equations. Nonlinear equations are equations where the unknown quantity that we want to solve for appears in a nonlinear fashion. For example, if the quantity in question is a function y(t), then terms such as  $y^2$ ,  $y\frac{dy}{dt}$  or  $\sin y$  etc. would be nonlinear. More precisely, a nonlinear equation is one where a linear combination of solutions is not a new solution. If we denote  $\dot{y} = \frac{dy}{dt}$ , then the equation

$$\dot{y} = y(y-1),$$
 (1.1)

is clearly nonlinear because of the  $y^2$ -term.

A nonlinear system may depend on one or several parameters (such as t in our example).

There are two main types of nonlinear systems: *differential equations* and *iterated maps*. You are already familiar with differential equations from previous years and equation (1.1) is an example of this type. In this module we will concentrate on nonlinear differential equations (although before that we will review a lot of results for linear equations!).

Differential equations describe nonlinear systems where the unknown function or functions depend on continuous variables (such as  $t \in \mathbb{R}$  in (1.1)). In contrast, iterated maps arise in problems where the variables on which the solutions of the system depend take only discrete values. For example the equation:

$$x_{n+1} = x_n(x_n - 1),$$
 with  $n = 0, 1, 2...$  (1.2)

is an iterated map, since it depends on a discrete variable n. If we know the value of  $x_0$  we can obtain  $x_1$  by substituting  $x_0$  on the r.h.s. of (1.2). Once we know  $x_1$  we can find out  $x_2$  by repeating the same "iterated" process.

A dynamical system is the same as a nonlinear system where the nonlinear equations represent the evolution of a solution with time or some variable that may be interpreted as time. For example, if we look at our equations (1.1) and (1.2) and interpret the variables t and n as time respectively, then we can think of these two equations as describing a dynamical system. Indeed, if we just think of the

word dynamical, it suggests to us movement or change in time. The name dynamical originated in the context of physics, where nonlinear equations are very common.

Dynamical systems and nonlinear equations describe a great variety of phenomena, not only in physics, but also in economics, finance, chemistry and biology.

In this module we will mostly concentrate in learning the mathematical techniques that allow us to study and classify the solutions of dynamical systems. When possible, we will also emphasize the context in which the equations arise, namely, their interpretation.

Another way of defining a dynamical system is to say that it consists of a set of things called *States* and some *Rule* that takes us from one state to another in some prescribed way. In the examples (1.1) and (1.2) we can think of a "state" as the solution to the equations at a given time (e.g. a fixed value of t or n). Once we know such solution, we can use the equations to find out what the solution is for a later (or earlier) time.

Nonlinear equations are generally much harder to solve that linear ones. Most techniques for solving differential equations which you have seen in the last two years only really hold for linear equations. In fact, for many nonlinear equations it is not possible to find an explicit, analytic solution. They can only be solved numerically.

The additional complexity of nonlinear equations often leads to extremely interesting behaviour, such as the appearance of chaos and the relationship between nonlinear equations and fractals. For example, the fractal that you can find at http://en.wikipedia.org/wiki/Mandelbrot\_set is called the Mandelbrot set and its picture can be generated by iterating the simple map  $x_{n+1} = x_n^2 + c$ , where c is a constant.

In this first chapter I will introduce by means of examples some of the main definitions/techniques that are used to study dynamical systems: the definition of phase space and how to draw phase space diagrams.

#### **Example 1.1** - State of a system (1 dimensional)

Consider a population p governed by the differential equation

$$\frac{dp}{dt} = p(1-p),\tag{1.3}$$

where t represents time<sup>1</sup>. Given the population at some time  $t_0$  then the equation can be used to predict the population at some later (or even earlier) time t. This problem is one dimensional with a single state variable p, the rule that tell us how to move to another state is the differential equation Eq 1.3. Thus the time variable t is just a parameter that traces a path from one state to another. The set of all states is called the **phase space** or *state space*. A diagram of all the states indicating a direction corresponding to increasing t (the independent variable) is called a **phase diagram**.

The phase diagram for this problem is given in Fig 1.1 where we have only shown the region  $p \ge 0$ 

$$\frac{dp}{dt} = p$$

<sup>&</sup>lt;sup>1</sup>This is in fact a very famous equation known as Logistic Equation. It provides the most simple realistic model for a population, as it accounts both for deaths and births. If we consider the simpler linear equation

we will quickly see that it is solved by  $p(t) = p_0 e^t$ , where  $p_0$  represents the initial population. Such as solution represents a population that grows exponentially in time. Any population with infinite resources and no predators would do so, but in most cases resources are limited and there are also predators, so it is sensible to modify the equation to account for this. This is achieved by introducing the term  $-p^2$  on the r.h.s. This term represents a kind of competition between individuals of the same population which tends to decrease the population in time.





since the system refers to a population which cannot realistically be negative. However a complete analysis of the system may well include all values of p. The direction of the arrows have been assigned to the diagram using the fact that if the derivative of a function is positive then the function is increasing with respect to its variable and similarly if the derivative is negative it is decreasing. Additionally if the derivative is zero then the function is said to be stationary. At such points we have constant (or fixed point) solutions. Indeed p = 0 and p = 1 are constant solutions to the problem.

To sketch the diagram:

• First identify and insert the stationary (fixed)points as follows.

$$\frac{dp}{dt} = 0 \quad \Rightarrow \quad p(1-p) = 0 \quad \Rightarrow \quad p = 0, \quad p = 1$$

Thus we draw a line with the two points p = 0 and p = 1 indicated. (usually with a full dot)

- In the region  $0 we can see that <math>\frac{dp}{dt} = p(1-p) > 0$  thus p is increasing in this region. Hence the direction of the arrow in this region is from left to right.
- In the region 1 < p we can see that  $\frac{dp}{dt} = p(1-p) < 0$  thus p is decreasing in this region. Hence the direction of the arrow in this region is reversed.

The diagram reveals, without any further attempt to solve Eq 1.3, that for any initial population the long term trend is for the population to either increase or decrease to p = 1. This is an example of a *limited growth* model and the point p = 1 is called an *attractor* (we will see a proper definition of this later on). The interpretation of this is that if the initial population is small it will grow until a maximum is reached, which corresponds to an equilibrium between the amount of births and deaths. On the other hand, if the starting population is too large, they will compete with each other for resources, which will be scarce, so that for a while deaths will dominate births, until again an equilibrium is reached.

As you can see, one can actually say quite a lot about the solutions to a differential equation without necessarily solving it!

In this case, it is actually very easy to solve the equation, since we can completely separate variables p(t) and t

$$\frac{dp}{dt} = p(1-p) \quad \Leftrightarrow \quad \frac{dp}{p(1-p)} = dt \quad \Leftrightarrow \quad \int \frac{dp}{p(1-p)} = \int dt = t+c, \tag{1.4}$$

where c is an integration constant. Using partial fractions we have

$$\int \frac{dp}{p(1-p)} = \int \left(\frac{1}{p} + \frac{1}{1-p}\right) dp = \log p - \log(1-p) = \log \frac{p}{1-p}.$$
(1.5)

Therefore

$$\log \frac{p}{1-p} = t + t_0 \quad \Leftrightarrow \quad \frac{p}{1-p} = e^{t+c}, \tag{1.6}$$

which gives

$$p(t) = \frac{1}{1 + (p_0^{-1} - 1)e^{-t}},$$
(1.7)

where  $p_0$  is the population at time t = 0 which is given by  $p_0 = e^c/(1 + e^c)$  in terms of the original integration constant.



Figure 1.2: The figure on the l.h.s. represents the solution p(t) for  $p_0 = 1/2$ . As expected p(t) increases with time in this case until reaching the maximum value of 1. The figure on the r.h.s. corresponds to initial condition  $p_0 = 2$ . In this case the behavior is the opposite. The function p(t) decreases with time until reaching the asymptotic value 1.

Let us introduce the following definition:

# Definition

The equation

$$\frac{dy}{dx} = X(y)$$

is said to have a **fixed point** at y = a if and only if X(a) = 0. Clearly y = a is also a solution of the equation.

Fixed points are also referred to as **critical points**.

#### **Example 1.2** - 2-dimensional problem - Competition

Consider the following two dimensional problem that is used to model the advertising budgets of two

competing companies<sup>2</sup>; company X spending x per year and company Y spending y per year.

$$\frac{dx}{dt} = x(a-x) + by \tag{1.8}$$

$$\frac{dy}{dt} = y(c-y) + dx \tag{1.9}$$

The equations are of course an over simplification however they contain some of the basic ideas of modelling. We note that in the absence of the Y company the advertising budget of the X company is given by Eq1.8 with b set equal to zero. This is precisely the type of problem we had for the single population p in Example 1.1. Thus for a single company the advertising budget would stabilise. The introduction of the by term in Eq(1.8) has the tendency to increase the rate of change of x. This is as one would expect, the X company will start to spend more on advertising due to the appearance of a competitor in the market. Similar remarks apply to the Y company.

The question we now ask is what happens to the two budgets as time evolves and what are the affects of a, b, c and d on this behaviour. This question is quite difficult to answer, however theoretically we could produce a phase diagram in two dimensions, that is a plot of y against x, by carrying out the following steps:

By dividing Eq(1.9) by Eq(1.8) and using 
$$\frac{dy}{dt} = \frac{dy}{dx}$$
 we obtain:  

$$\frac{dy}{dx} = \frac{y(c-y) + dx}{x(a-x) + by}$$
(1.10)

Unfortunately this first order equation is quite difficult to solve. To see in detail how this works consider the following simple example.

**Example 1.3** - Phase diagram for a simple 2-dimensional system

We now consider a system whose phase diagram can be obtained using the method described in Ex(1.2).

Given:

$$\frac{dy}{dt} = -x \tag{1.11}$$

$$\frac{dx}{dt} = y \tag{1.12}$$

and applying the above method gives:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{x}{y}$$

<sup>&</sup>lt;sup>2</sup>It can also be interpreted as modelling competition between two populations. In that case it is a generalization of the equation of the first example to the case a two competing species. Now the population x is not only affected by competition amongst its members but also by the presence of y.

Hence:

$$\frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad \int y \, dy = \int -x \, dx \quad \Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C \quad \Rightarrow \quad y^2 + x^2 = 2C$$

If we now consider the phase path through the point P given by x = 1, y = 2 then C = 5/2 and the phase path is given by  $x^2 + y^2 = 5$  which is a circle centered at the origin with radius  $\sqrt{5}$  (See Fig 1.3). The trajectories in the phase diagram should also indicate the direction of increasing t. We can do this by looking at either of the original equations Eq 1.11 or Eq 1.12. Considering Eq 1.12 at P, we see that since y = 2,  $\frac{dx}{dt} = 2$ , which is positive. Thus x is increasing at P. This fact is indicated on the diagram Fig 1.3(a) by the direction of the arrow. The complete phase diagram consists of the



Figure 1.3: (a) Single phase path (trajectory) through P(1,2) (b) Phase diagram showing several such trajectories The arrows indicate the direction of increasing t.

set of all concentric circles centered at the origin. In other words, for any initial values of x and y we can always find a circle (a value of the constant C) that contains the point (x, y). See Fig 1.3(b).

There are quite a lot of things that we can say about the solutions to equations (1.11)-(1.12) as functions of t, just by looking at the phase diagram. Since all trajectories are circles we can see that both x(t) and y(t) are periodic function of t. The reason is that no matter where we start in the circle, as time progresses we will always eventually come back to the same point. We also see that whenever x reaches its maximum or minimum value we have y = 0 and, similarly, y is maximum or minimum when x = 0. This behaviour clearly reminds us of the periodic functions sine and cosine.

Indeed the solutions to equations (1.11)-(1.12) are

$$x(t) = A\sin t, \qquad \qquad y(t) = A\cos t, \qquad (1.13)$$

where A is a constant that is fixed by the initial conditions. Clearly  $x^2 + y^2 = A^2 = \text{constant}$  as we have already seen from our phase space analysis. In order for the point P = (1, 2) to be part of the phase space trajectory we need  $A^2 = 2C = 5$ , which fixes  $A = \sqrt{5}$ . The value of t for which the solutions x(t) and y(t) pass by the point (1, 2) can be obtained by solving:

$$\sqrt{5}\sin t = 1 \qquad \sqrt{5}\cos t = 2,$$

which gives

$$\tan t = \frac{1}{2},$$

that is t = 0.463648.

## Example 1.4 - Second Order Differential System

We consider now a simple problem of a pendulum swinging about a smooth pivot. To simplify things the only mass in the problem is that of the bob at the end of the pendulum arm; the arm itself is considered to be rigid and have negligible mass compared with that of the bob. Gravity is considered to be the only force in the problem; friction due to the pivot and air resistance are ignored. (see Fig 1.4) The equation governing the motion of the pendulum can be written as:

$$\frac{d^2\theta}{dt^2} = -\sin\theta \tag{1.14}$$

where t represents time.

The position of the pendulum is clearly given by the value of  $\theta$ , however this alone is not sufficient to determine the state of the system. We would not say that a stationary pendulum with  $\theta = \pi/4$ , say, was in the same state as a moving pendulum with  $\theta = \pi/4$ . We thus need to introduce the speed at which the pendulum is swinging to completely describe the state of the system. This makes the problem a two dimensional system with two suitable state variable being  $\theta$  for position and  $\frac{d\theta}{dt}$  for speed.

To make this problem fit into the general first order approach we set  $x = \theta$  and  $y = \dot{\theta}$ . The single second order equation Eq(1.14) can then be written as two first order equations as follows:

$$\frac{dx}{dt} = \frac{d\theta}{dt} = y \quad \text{and} \quad \frac{dy}{dt} = \frac{d^2\theta}{dt^2} = -\sin\theta = -\sin x \quad \Rightarrow$$
$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -\sin x \tag{1.15}$$

Dividing the equations in Eq(1.15) gives:

$$y\frac{dy}{dx} = -\sin x \quad \Rightarrow \quad \int y \, dy = -\int \sin x \, dx \quad \Rightarrow \quad y^2 - 2\cos x = C$$
(1.16)

The phase diagram for the system therefore consists of a two dimensional plot of y against x given by Eq(1.16). Although quite difficult to produce the details are given in Fig 1.5

With reference to the phase diagram, if we trace the path of the point P as time evolves, P will in fact move round the path in a clockwise direction. At P,  $x = \theta = -\pi/2$  and  $y = \dot{\theta} = 1$ , which represents the pendulum at the three o'clock position moving in a clockwise direction. Other points on this curve will represent other positions and speeds of the pendulum. At the point  $(-\pi/2, 1)$  the integration constant in equation (1.16) gets fixed to C = 1 and the phase space trajectory is a closed trajectory corresponding to joining together the two functions:

$$y = \pm \sqrt{1 + 2\cos x}$$



Figure 1.4: The pendulum swings under the influence of gravity, no other forces, such as friction, are considered. The angle theta is measured as positive in the clockwise direction

Each of these trajectories ends when y = 0, which in this case corresponds to  $x = \pm 2\pi/3$ . For larger values of x the function under the square root becomes negative and therefore there is no solution y for such values of x. As in the previous example, when we had circular trajectories in phase space, we find once again that if our system starts at initial conditions corresponding to point P in the figure, it will eventually return to the same point (that is the original position and angular speed), representing a cyclic or swinging motion of the pendulum<sup>3</sup>

In contrast if the system starts at the point Q,  $\theta$  does not return to it original value, indeed it continues to increase. This will represent a start up situation where the pendulum is given a sufficiently large value of  $\dot{\theta}$  (*hit really hard to set it in motion*) to make it continually rotate about its pivot rather than just swing. In terms of the function (1.16) what this means is that the integration constant C is now sufficiently large so that the function y never becomes zero. An example would be to take again  $x = \theta = -\pi/2$  and  $y = \dot{\theta} = 3$ . In this case C = 3 and we have again two solutions:

$$y = \pm \sqrt{3} + 2\cos x.$$

Now the function under the square root is never zero and never negative. This means that in Fig. 1.5 the two solutions for y (with the plus and minus sign) do not meet anymore. Physically this means that if the pendulum starts with positive velocity  $\dot{\theta}$  (clockwise direction) it will always have positive velocity (it will always move in the clockwise direction). If it starts with negative velocity it will always have negative velocity (anticlockwise direction).

Although the trajectories do not seem closed anymore, there is still periodicity in the sense that if we keep drawing our solution for larger or smaller values of x it will keep repeating itself (it oscillates periodically).

The phase diagram again gives us qualitative information regarding the system without actually solving the equation fully. Indeed obtaining  $\theta$  in terms of t is not possible using elementary functions

<sup>&</sup>lt;sup>3</sup>You will see that the smaller the initial value of  $y = \dot{\theta}$ , the more the trajectories look like a perfect circle. This can be explained by noticing that for  $x = \theta$  small, the original equation can be approximated by approximating the sine function as  $\sin \theta \approx \theta$  (1st term in the Taylor expansion about  $\theta = 0$ ). If we do this the system of equations (1.15) becomes exactly the same as in the previous example.

 $\frac{dx}{dt} = y$ 



x increasing when y > 0

Figure 1.5: Phase diagram for  $\ddot{\theta} = -\sin\theta$ ; arrows indicate direction of increasing t

# **General Result**

Given:

$$\frac{dx}{dt} = X_1(x,y)$$
 and  $\frac{dy}{dt} = X_2(x,y)$ 

the phase diagram for this system can be constructed by solving the first order equation:

$$\frac{dy}{dx} = \frac{X_2(x,y)}{X_1(x,y)}$$

Whether we are considering a one dimensional system given by a single first order differential equation or whether we are considering a genuine second order system it will be essential that we can solve simple first order differential equations. The next section reviews two of the most common methods.

# **1.2** First Order Differential Equations

This section is split into three parts, namely the definition of what is meant by a solution to a differential equation, how to obtain a rough sketch of the solution and finally how to solve exactly two type of first order differential equation.

# 1.2.1 Definition of Solution

# Definition

Let X(x, y) be a real valued continuous function of the real variables x and y defined in some open <sup>4</sup> domain D of  $\mathbb{R}^2$ . A continuously differentiable function y(x) with x in some interval I such that

 $<sup>^4\</sup>mathrm{In}\ \mathbb{R}^2$  this can be thought of as an area that does not include its boundary

## CHAPTER 1. INTRODUCTION

$$\frac{dy(x)}{dx} = X(x, y(x)) \text{ is said to be a solution of the equation } \frac{dy}{dx} = X(x, y).$$

Perhaps one of the most important facts to note is that even if X(x, y) is a reasonable function a solution to the differential equation may, if it exists, only be defined on some interval of  $\mathbb{R}$  rather than for all values of x. This definition always assumes that we are using the maximum possible interval when considering a solution.

As an example of this definition consider the problem solved in Eg 1.3, namely:

$$\frac{dy}{dx} = -\frac{x}{y} \qquad x = 1, y = 2$$

- Here  $X(x,y) = -\frac{x}{y}$ , hence a suitable domain is any set excluding y = 0. Indeed the largest possible domain is the set  $D = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$ .
- As we saw the solution was  $y^2 + x^2 = 5$  hence  $y = \pm \sqrt{5 x^2}$  The definition of the solution requires that y(x) is a function, which means that given x we must only have one value of y. Thus we cannot have a  $\pm$  sign in the solution. Clearly  $y = +\sqrt{5 x^2}$ , since with the negative sign y will not equal +2 at x = 1.
- Since  $y = +\sqrt{5-x^2}$ , x is restricted to the range given by  $0 \le (5-x^2)$  in order for the square root to be valid. Since  $y \ne 0$ , x is restricted further to  $0 < (5-x^2)$ , that is to say,  $-\sqrt{5} < x < \sqrt{5}$ . The interval I therefore is given by  $I = (-\sqrt{5}, \sqrt{5})$ .

Thus graphically the solution to this problem is the part of the circle in Fig 1.3(a) above the x-axis.

#### Example 1.5

Consider:

$$\frac{dy}{dx} = -\frac{y}{x} \qquad y = 1 \quad \text{and} \quad x = 1 \tag{1.17}$$

Solving gives:

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \quad \Rightarrow \quad \ln y = -\ln x + C$$
$$\Rightarrow \quad \ln(xy) = C \quad \Rightarrow \quad xy = e^C = A$$

Applying the conditions x = 1, y = 1 gives the solution as  $y = \frac{1}{x}$ . The graph of  $y = \frac{1}{x}$  is in two halves, see Fig 1.6, however the only portion that gives a solution through (1,1) is the one defined on the interval  $I = (0, \infty)$ .

<sup>&</sup>lt;sup>5</sup>The open brackets ( ) are used to indicate that the end points of the interval are omitted. To include the end point we use the square brackets [ ]. eg (1,2) stands for the interval 1 < x < 2 and [1,2] for  $1 \le x \le 2$ 



Figure 1.6: The solution to Eq 1.17 is the portion in the positive quadrant defined on the interval  $I = (0, \infty)$ 

# 1.2.2 Solution by sketching

This is a method of obtaining a very rough sketch of the solutions to the equation  $\frac{dy}{dx} = X(x,y)$ . Recalling that  $\frac{dy}{dx}$  is the gradient of the tangent to y(x), we can use X(x,y) to construct a line segment with gradient equal to that of a solution through any considered point. Consider the lattice of points  $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1 \dots n\}$  and at each point construct a small line segment with gradient equal to the point. It therefore follows that each line segment is a small tangent to the solution through each point. Fig 1.7 shows the construction of such line segments



Figure 1.7: Velocity/vector field generated from dy/dx = -x/y. The trajectories can clearly be seen to be concentric circles about the origin as in Fig 1.3

for the system considered in Example 1.3 and illustrated in Fig 1.3. It is clear that from Fig 1.7 it is possible to sketch freehand the trajectories, ensuring at all times that the curves run in the same direction as nearby line segments. A good strategy for constructing the sketch is to start at the edge of the diagram between two line segments, drawing each curve with a slope somewhere between the

slopes of each pair of segments.

Producing a diagram such as that on Fig 1.7 is a rather painstaking process and therefore is not very practical to do this by hand. That is why I will be providing you with an Excel Macro that is going to do such drawings for you (see the module's web-site).

### **1.2.3** Exact solution of first order differential equations

In this section we consider two elementary methods for solving first order differential equations. Namely the methods of **separation of variable** and **integrating factor**. As we shall see these can only be applied to a restricted set of equations but nevertheless prove to be extremely useful.

**Separation of Variable** We have already encountered this method above in Example1.1 and Example 1.3. The general result is given by:

$$\frac{dy}{dx} = h(x)g(y) \quad \text{then} \quad \int \frac{dy}{g(y)} = \int h(x) \, dx \tag{1.18}$$

Clearly to obtain an exact solution to the problem it is now necessary to carry out the integration, this we know may not be a trivial task.

### Example 1.6

Given

$$\frac{dy}{dx} = x \tan y \qquad D = \{(x, y) \in \mathbb{R}^2 : y \neq (2n+1)\pi/2, \ n = 0, \pm 1, \pm 2\dots\}$$

find the solution such that  $y = \pi/4$  at x = 0, carefully specifying its interval I of definition. Since the righthand side of the equation is the product of two functions, one in x and one in y we use the method of separation of variable to give:

$$\int \frac{dy}{\tan y} = \int x \, dx \quad \Rightarrow \quad \ln(\sin y) = \frac{x^2}{2} + C \quad \Rightarrow \quad y = \sin^{-1}(Ae^{x^2/2})$$

The condition  $y = \pi/4$  at x = 0 gives the solution  $y = \sin^{-1}\left(\frac{e^{x^2/2}}{\sqrt{2}}\right)$  where we have assumed the principal value of  $\sin^{-1}$  in order to give  $y = \pi/4$  at x = 0.

We now ask the question: is this expression valid for all values of x? The function  $\sin^{-1}$  has domain equal to the interval [-1,1] which implies that:

$$-1 \le \frac{e^{x^2/2}}{\sqrt{2}} \le 1 \quad \Leftrightarrow \quad \frac{e^{x^2/2}}{\sqrt{2}} \le 1 \quad \Leftrightarrow \quad e^{x^2/2} \le \sqrt{2} \quad \Leftrightarrow \quad x^2 \le 2\ln\sqrt{2} = \ln 2$$

Hence:

$$-\sqrt{\ln 2} \le x \le \sqrt{\ln 2}$$

Finally we note that since  $y = \pm \pi/2$  is not in the domain of the problem we need to exclude the possibility of  $x = \pm \sqrt{\ln 2}$ .



Figure 1.8:

Thus  $I = (-\sqrt{\ln 2}, +\sqrt{\ln 2})$ . See Fig 1.8

## **Integrating factor**

We now consider the first order linear differential equation in the standard form:  $^{6}$ 

$$\frac{dy(x)}{dx} + P(x)y(x) = Q(x)$$
(1.19)

From Elementary Calculus we have that if we multiply this equation by  $R(x) = \exp(\int P(x) dx)$  then we obtain:

$$\frac{d R(x)y(x)}{dx} = R(x)Q(x)$$

Hence

$$R(x)y(x) = \int R(x)Q(x) \, dx + C \quad \Rightarrow \quad y(x) = \frac{1}{R(x)} \left\{ \int R(x)Q(x) \, dx + C \right\}$$

After carrying out the integration the arbitrary constant C is calculated using a given initial condition such as  $y = y_0$  at  $x = x_0$  that is to say  $y(x_0) = y_0$ .

An **alternative** method of automatically including the initial condition in the calculation rather than applying it later in order to determine C is to carry out a definite integral with the lower limit equal to  $x_0$  and the upper limit set to the x. Doing this we obtain:

$$\frac{d R(x)y}{dx} = R(x)Q(x) \quad \Rightarrow \quad \int_{x_0}^x \frac{d R(x)y}{dx} \, dx = \int_{x_0}^x R(x)Q(x) \, dx$$

Hence

$$[R(x)y(x)]_{x_0}^x = \int_{x_0}^x R(x)Q(x) \, dx \quad \Rightarrow \quad R(x)y(x) - R(x_0)y_0 = \int_{x_0}^x R(x)Q(x) \, dx$$

Thus

$$y(x) = \frac{1}{R(x)} \left\{ R(x_0)y_0 + \int_{x_0}^x R(x)Q(x) \, dx \right\}$$

<sup>&</sup>lt;sup>6</sup>to emphasise the dependence on x we have denoted y as y(x)

The following example will be used to demonstrate both these methods;

## Example 1.7

Given  $y = \pi$  at  $x = \pi$  solve the following differential equation:

$$\frac{dy}{dx} - \frac{y}{x} = x\cos x \qquad x \neq 0 \tag{1.20}$$

Calculating the integrating factor:

$$R(x) = \exp\left\{\int -\frac{1}{x} dx\right\} = \exp\{-\ln x\} = \frac{1}{x}$$

Thus multiplying Eq 1.20 by R(x) gives:

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{1}{x}x\cos x = \cos x$$

Thus

$$\frac{y}{x} = \int \cos x \, dx + C \quad \Rightarrow \quad y = x \sin x + Cx$$

The condition  $y(\pi) = \pi$  implies that C = 1, thus the solution is given by  $y = x(\sin x + 1)$ . It appears that this solution is defined for all x, however the domain does not contain x = 0, thus for the solution to be defined on an interval that excludes x = 0 but includes  $x = \pi$  we are restricted to  $I = (0, \infty)$ .

Using the **alternative** method:

$$\int_{\pi}^{x} \frac{d}{dx} \left(\frac{1}{x} y\right) dx = \int_{\pi}^{x} \cos x \, dx \quad \Rightarrow \quad \left[\frac{y(x)}{x}\right]_{\pi}^{x} = [\sin x]_{\pi}^{x}$$

Thus

$$\frac{y}{x} - \frac{\pi}{\pi} = \sin x - \sin \pi \quad \Rightarrow \quad y = x(\sin x + 1)$$

which is the same result as before.

# 1.2.4 Existence and uniqueness

We end this section with a short statement regarding the existence and uniqueness of our solution. Given an equation solutions may not always exist and if they do there may be more than one solution satisfying a given initial condition. A first order differential equation with the condition  $y(x_0) = y_0$  is referred to as an **initial value problem**. The following theorem provides a set of sufficient conditions for the initial value problem to have a unique solution. The conditions of the theorem are not necessary, that is to say there may well be equations with unique solutions that do not satisfy all the conditions.

### Theorem

Let X(x, y) be defined on some open <sup>7</sup> domain  $D \subseteq \mathbb{R}^2$  such that both X(x, y) and  $\frac{\partial X(x, y)}{\partial y}$  are continuous on D with the point  $(x_0, y_0) \in D$ . In such a case there exists a unique solution y(x) of  $\frac{dy}{dx} = X(x, y)$  defined on some interval I with  $x_0 \in I$  such that  $y(x_0) = y_0$ .

We will not pursue this further but always assume that there does exist a unique solution to all our problems. As one would expect this theorem can be extended into higher dimensions.

<sup>&</sup>lt;sup>7</sup>Roughly speaking we should think of an open set in  $\mathbb{R}^2$  to be a region of  $\mathbb{R}^2$  without its boundary eg  $x^2 + y^2 < 1$  is the open set represented by the interior of a unit circle centred at the origin

# Chapter 2

# First Order Autonomous Dynamical Systems

# 2.1 Introduction

In chapter 1 we saw that a first order differential equation could either be used to describe a one dimensional problem such as the population model or be used to create a phase diagram for a two dimensional system as found in Example 1.3. In this chapter we look at the one dimensional systems in their own right and concentrate our attention to systems that do not depend explicitly on the independent variable. Such systems are referred to as autonomous.

#### Definition

The equation  $\frac{dy}{dx} = X(y)$  is said to be autonomous as its righthand side does not depend explicitly on x.

## Example 2.1

Consider the equation:

$$\frac{1}{k}\frac{dy}{dx} = a^2 - y^2 \qquad k > 0 \tag{2.1}$$

This equation can be used to model the free fall of an object through the air taking into account gravity and air resistance. In such a case y would stand for the speed of the object and x time of fall. Before trying to solve the equation we will construct its phase diagram by identifying regions where y is increasing , decreasing or neither. From such a diagram we are able to make certain qualitative statement about the system, such as given the value of y at some value of x what happens to y as x increases. The system is one dimensional thus the phase diagram is just a straight line. The phase

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Figure 2.1: a is assumed to be positive. The three regions y < -a, -a < y < a and y > a are labelled (1), (2) and (3) respectively.

diagram Fig 2.1 is constructed as follows:

- mark on the diagram the points where y is stationary. This occurs where  $\frac{1}{k}\frac{dy}{dx} = a^2 y^2 = 0$ . Thus we mark y = +a and y = -a
- In region (1) y < -a thus dy/dx < 0 and hence y is decreasing with respect to x. This is indicated on the diagram by the first arrow.
- In region (2) -a < y < a thus dy/dx > 0 and hence y is increasing with respect to x. Again this is indicated using an arrow
- Similarly in region (3) y is decreasing which again is indicated by the arrow.

Fig 2.1 is the phase diagram for the system. We see that if the system starts with any value of y greater than -a the system will tend to the fixed value y = a, either monotonically increasing or decreasing to this value as appropriate. With an initial value of y less than -a the system moves away from -a towards  $-\infty$ . The precise interpretation of the diagram will always depend on the physical situation being modelled. For the falling object problem the equation is actually only valid for  $y \ge 0$  thus the fact that all starting values greater or equal to zero tend to +a is interpreted physically as the terminal velocity of a falling object. ie all falling objects reach a maximum speed known as the terminal velocity.

The phase diagram can also be helpful in sorting out the different types of solutions obtained by starting in each of the three regions. In this problem we can actually obtain the exact solution to the differential equation, though in most realistic problems this is not possible and we need to resort to a numerical method. In such a case the phase diagram is then useful in providing us with prior knowledge of the solution.

### **Exact solution**

Using the general starting condition  $y(x_0) = y_0$  and the method of separation of variables, integrating Eq 2.1 gives:

$$\int_{y_0}^{y} \frac{dy}{a^2 - y^2} = \int_{x_0}^{x} k \, dx \quad \Rightarrow \quad \frac{1}{2a} \int_{y_0}^{y} \left(\frac{1}{a - y} + \frac{1}{a + y}\right) \, dy = \int_{x_0}^{x} k \, dx$$

Which on integration gives:

$$\left[\frac{1}{2a}\ln\left(\frac{a+y}{a-y}\right)\right]_{y_0}^y = [kx]_{x_0}^x$$

Hence

$$\ln\left(\frac{a+y}{a-y}\right) - \ln\left(\frac{a+y_0}{a-y_0}\right) = 2ka(x-x_0)$$
$$\Rightarrow \quad \ln\left\{\left(\frac{a+y}{a-y}\right)\left(\frac{a-y_0}{a+y_0}\right)\right\} = 2ka(x-x_0)$$

$$\Rightarrow \quad \frac{a+y}{a-y} = \left(\frac{a+y_0}{a-y_0}\right) e^{2ka(x-x_0)}$$

After some manipulation this gives:

$$y = a \frac{\left\{ (a+y_0)e^{2ka(x-x_0)} - (a-y_0) \right\}}{\left\{ (a+y_0)e^{2ka(x-x_0)} + (a-y_0) \right\}}$$
(2.2)

Our task now is to plot the solution obtained in Eq 2.2 for values of  $y_0$  in each of the three regions identified in Fig 2.1.

First however note the following:

- The solution depends on  $(x x_0)$ , ie the difference between x and its starting value. If in an example x represented time, the solution is seen to only depend on the amount of time elapsed from the beginning and not the actual time on your clock or date on your calendar. This property is a consequence of the problem being autonomous, indeed it can be proved that all solutions of an autonomous system can be expressed in terms of  $(x x_0)$ . Graphically this means that all the solutions will *essentially* be the same; in each of the three regions any one solution is a translation of the other.
- With k and a assumed to be greater than zero the formula in Eq(2.2) implies that y tends to +a as x tends to infinity and to -a as x tends to minus infinity for all values of  $y_0$ , with of course the exception of  $y_0 = \pm a$ , which are constant solutions. However as seen below since a solution of a differential equation can only be defined on an interval it may not be meaningful to carry out these limits for some starting values  $y_0$ .

Graph in region (2)  $-a < y_0 < +a$ 

With  $y_0$  in this range the denominator of the solution in Eq 2.2 can never be zero as it consists of the sum of two positive terms for all values of x. Thus the interval of definition in this range is given by  $I = (-\infty, +\infty)$ . The graph is given in Fig 2.2; we note that the solution has the expected behaviour as  $x \to \pm \infty$ .



Figure 2.2: Solution of Eq 2.2 for  $-a < y_0 < a$ ;  $I = (-\infty, +\infty)$ 

## Graph in region (3) $a < y_0$

In this region  $(a - y_0)$  is negative and  $(a + y_0)$  is positive, hence the denominator of the solution in Eq 2.2 will vanish at some value of x. Denoting this value of x as  $x_{\infty}$  it is given by:

$$(a+y_0)e^{2ka(x_{\infty}-x_0)} + (a-y_0) = 0$$

Solving for  $x_{\infty}$  this gives:

$$x_{\infty} = x_0 + \frac{1}{2ak} \ln\left(\frac{y_0 - a}{y_0 + a}\right)$$
(2.3)

Also in this range we observe that  $y_0 - a < y_0 + a$  thus  $\frac{y_0 - a}{y_0 + a} < 1$  which means that the ln term in Eq. 2.3 is negative and hence we can deduce that  $x_{\infty}$  is to the left of  $x_0$ . The interval of definition can not include  $x_{\infty}$  but must include  $x_0$ , therefore  $I = (x_{\infty}, \infty)$ .(see Fig 2.3) With this interval of definition it is not meaningful to consider the limit of Eq(2.2) as  $x \to -\infty$ . Indeed as x decreases from  $x_0$  towards  $x_{\infty}$  the solution y tends to  $+\infty$  and not -a. The limit of Eq(2.2) as  $x \to +\infty$  is meaningful. As expected, and as can be seen in Fig.(2.3), y tends to +a as x tends to  $\infty$ .



Figure 2.3: Solution of Eq 2.2 for  $a < y_0$ ;  $I = (x_{\infty}, \infty)$ 

# Graph in region (1) $y_0 < -a$

In this region  $(y_0 + a)$  is negative and  $(a - y_0)$  is positive, hence the denominator of the solution in Eq 2.2 will again vanish for some value of x. As before, denoting it as  $x_{\infty}$ , it is given by Eq 2.3. However this time  $\frac{y_0 - a}{y_0 + a} > 1$  and hence the ln term in Eq 2.3 will be positive and hence  $x_{\infty}$  will be to the right of  $x_0$ . In this case the interval of definition is given by  $I = (-\infty, x_{\infty})$  See Fig 2.4.



Figure 2.4: Solution of Eq 2.2 for  $y_0 < -a$ ;  $I = (-\infty, x_\infty)$ 

We now make the following observations:

- The starting value  $y_0$  clearly influences the solution greatly, changing its shape and interval of definition.
- The solution depends on  $(x x_0)$  which means that the solutions do not change shape as we vary  $x_0$  but simply translate to the right or left. This property is common to all autonomous systems. Physically, treating x as time, it is saying that the system performs in an identical manner no matter when you start.
- The points y = -a and y = +a are special to this problem. Not only are they the values of y at which  $\frac{dy}{dx} = 0$  but they are also constant (fixed) solutions of the equation. Clearly the original equation Eq 2.1 is satisfied by substituting  $y = \pm a$  into both sides.



Figure 2.5: Complete Solution of Eq 2.2 with  $y_0$  shown in each of the three regions.

# 2.2 Classification of Fixed Points

In the introduction, Sec.2.1, we considered an example which demonstrated two different types of fixed point. The one at y = a attracted solutions, in this sense it is considered as stable. The other fixed point at y = -a repelled solutions and will be thought of as unstable. To consolidate these ideas we make the following definitions.

### **Definition - stable**

A fixed point  $P^{1}$  is stable if all trajectories<sup>2</sup> that start close to P stay close to P as x increases.

Additionally a fixed point is **asymptotically stable** if all trajectories close to P tend to P as  $x \to \infty$ .

#### Definition - unstable

A fixed point is unstable if it is not stable.

The above definitions are not rigorous but they are applicable to fixed points of any dimensional autonomous system.

#### **Definition - attractor**

For a one dimensional system an asymptotically stable fixed point is called an **attractor**.

### **Definition - repellor**

For a one dimensional system a fixed point P such that all trajectories close to P move away from P, as x increases, is called a **repellor**.

### **Definition - shunt**

For a one dimensional system a fixed point P such that in every neighbourhood of P some trajectories are attracted to P and some are repelled by P is called a **shunt** 

In the final example of Sec. 2.1 the fixed point at y = +a is an attractor and is therefore stable; the fixed point at y = -a is a repellor and therefore unstable.

## Example 2.2

Obtain and classify the fixed points of  $\frac{dy}{dx} = y^2(y^2 - 1)$ . Hence draw the phase diagram.

- The fixed points are located by solving  $y^2(y^2 1) = 0$  which gives y = 0, y = +1 and y = -1. Enter these onto the phase diagram in Fig 2.6. This splits the phase diagram into four distinct regions.
- In region (1)  $dy/dx = y^2(y^2 1) > 0$ , thus y is increasing. This is indicated by the direction of the arrow from left to right.
- In regions (2) and (3)  $dy/dx = y^2(y^2 1) < 0$ , thus y is decreasing in both these regions. This is indicated by the direction of the arrows from left to right.
- In region (4)  $dy/dx = y^2(y^2 1) > 0$ , thus y is again increasing and indicated by the direction of the arrow.

Thus we see from the direction of the arrows that y = -1 is an **attractor** and therefore **stable**; y = +1 is a **repellor** and therefore **unstable**. The fixed point at y = 0 is a mixture of attractor and

 $<sup>^{1}</sup>$ P is a point in the phase plane; in 1-dimension P is a point on the straight line, in two dimensions it is a point in a plane.

 $<sup>^{2}</sup>$ a trajectory is a path in the phase plane; in 1-dimension a trajectory is a part of the real line, in 2-dimensions it is a curve, both depend on the independent variable in the problem.



Figure 2.6: Phase diagram for  $dy/dx = y^2(y^2 - 1)$ 

repellor in all its neighbourhoods, therefore it is a **shunt**. A **shunt is not stable** since not all trajectories close to it stay close. Indeed as we see, if y is to the left of 0 it will move way from 0 towards  $-\infty$ .

Additionally we remark that y = 0, y = 1 and y = -1 are solutions of the equation.

# 2.3 Linearisation of a 1-dimensional systems

Classifying fixed points and obtaining solutions to a given problem can in general be quite difficult. We look now at a method that enables us to consider the system in the neighbourhood of its fixed points. By considering a simplified version of the problem in each of these neighbourhoods we can classify the fixed points and obtain approximations to the solutions.

#### 2.3.1 The Linear Equation

Consider the equation:

$$\frac{dy}{dx} = X(y) \tag{2.4}$$

and let y = a be a fixed point; ie X(a) = 0. Let us introduce a new variable z = y - a, that is y = z + a. We can now rewrite the original differential equation in terms of this new variable z as:

$$\frac{dz}{dx} = X(z+a). \tag{2.5}$$

Since we are interested in studying the behaviour of the system near the fixed point (y very close to a), this means that the values of z we are interested in are very small (since z = y - a). Therefore, it makes sense to expand X(z + a) using Taylor's expansion about z = 0.

Thus:

$$\frac{dz}{dx} = X(z+a) = X(a) + zX'(a) + \frac{z^2}{2}X''(a) + \dots$$

Since we wish to look at the problem close to y = a we will ignore terms in powers of  $z^2$  and above. Also since a is the fixed point X(a) = 0. Thus the linearisation of Eq 2.5 about y = a becomes:

$$\frac{dz}{dx} = zX'(a) \tag{2.6}$$

This differential equation has a solution which is defined for all x, that is to say the interval of definition for the linearised case is  $(-\infty, +\infty)$ . This may be different from the original problem where solutions may have only been defined on some interval I of the real line. The general solution to (2.6) is simply:

$$z = Ae^{X'(a)t}, (2.7)$$

where A is a constant that depends on the initial conditions of the problem.

Once we have the solution for z we can find the solution for y by simply using the definition y = z + a, so

$$y = a + Ae^{X'(a)t}. (2.8)$$

#### Example 2.3

As a first example consider the Eq. (2.1) at the beginning of this chapter with a = 2 and k = 1, namely,  $\frac{dy}{dx} = 4 - y^2$ . It has two fixed points at  $y = \pm 2$ . Consider now the linearisation of this equation about each of these points in turn. With reference to Eq. (2.5), in this example  $X(y) = 4 - y^2 \implies X'(y) = -2y$ . Thus at each of the two fixed points we have:

• About y = 2.

About this point y = 2 + z and the linearisation is given as:

 $z' = X(2)z \quad \Rightarrow \quad z' = -4z \quad \Rightarrow \quad z = Ae^{-4x} \quad \Rightarrow \quad y = 2 + Ae^{-4x}$ 

Introducing the initial condition  $y = y_0$  at  $x = x_0$  and calculating A gives:

 $y = 2 + (y_0 - 2)e^{-4(x - x_0)}$ 

Consider now the point P in Fig.(2.7). The curves that go by P represent the exact and linearised solutions to the equation for  $(y_0 - 2) > 0$ . In this region the linearised solution approaches 2 from above as  $x \to \infty$  and tend to  $+\infty$  as  $x \to -\infty$ . Note that as stated above this is defined for all x. Similar remarks apply if the linear solution is started in either of the other two regions.

• At y = -2

About this point y = -2 + z and the linearisation is given as:

$$z' = X(-2)z \quad \Rightarrow \quad z' = 4z \quad \Rightarrow \quad z = Ae^{4x} \quad \Rightarrow \quad y = -2 + Ae^{4x}$$

Introducing the initial condition  $y = y_0$  at  $x = x_0$  and calculating A gives:

$$y = -2 + (y_0 + 2)e^{4(x - x_0)}$$

We now consider this solution for the case  $y_0 < -a$ , the point Q in Fig.(2.7). As  $(y_0 + 2) < 0$  the solution approaches -2 from below as  $x \to -\infty$  and tend to  $-\infty$  as  $x \to +\infty$ .

Note that both these curves are shown plotted in Fig.(2.7). The important thing to note is that a linear solution is only close to an exact solutions when it is close to its associated fixed point. For example the linear solution about y = 2 through P is seen only to be close to the exact solution near y = 2.



Figure 2.7: Solutions of  $dy/dx = 4 - y^2$ ; 3 starting values for  $y(x_0) = y_0$  in the three regions  $y_0 < -2$ ,  $-2 < y_0 < 2$  and  $2 < y_0$ ; two linearisations, one about y = 2 initially passing through P and one about y = -2 initially passing through Q; Exact solution shown in thick line, linear approximations in thin line, asymptotes in dashed line.

# Example 2.4

As a second problem consider the population model from Chapter 1, namely  $\frac{dp}{dt} = p(1-p)$ . If all values of p are considered this has an attractor at p = 1 and a repellor at p = 0. The phase diagram is given by Fig 2.8 The phase diagram for this problem is the same as Fig 2.1, the phase diagram for the problem considered in Ex(2.3). If the differential equation is solved in the three regions p < 0, 0 and <math>1 < p, three different solutions are obtained. Again not all the solutions are defined for all values of t.

The solution satisfying the general initial condition  $p(t_0) = p_0$  is given by:

$$p = \frac{p_0 e^{(t-t_0)}}{(1-p_0) + p_0 e^{(t-t_0)}}$$

Again we see that the solution depends on  $(t - t_0)$ , a property of the equation being autonomous. Values of  $t_{\infty}$  are found by equating the denominator equal to zero for  $p_0$  in each of the three regions. The graph of all the solutions is given in Fig 2.9

Let us now consider the linearisation of this problem about its two fixed points. In this example



Figure 2.8: Phase diagram for dp/dt = p(1-p)

X(p) = p(1-p) thus X'(p) = 1-2p and hence X'(0) = 1 and X'(1) = -1. From the sign of X'(p) we can say immediately that p = 0 is a repellor and p = 1 is an attractor. We now look at the solutions of the linearised problem:

• About the point p = 0:

$$z' = X'(0)z = z \quad \Rightarrow \quad z = Ae^t$$

Given that  $p = p_0$  at  $t = t_0$  and remembering that in this case p = z, we get the linearised solution:

 $p = p_0 e^{(t-t_0)}$ 

This is plotted in Fig 2.9 with the initial point being at Q (R or P could equally well been chosen as the initial point to illustrate this solution). Note that the exact solution and this linear approximation get closer as p approaches zero.

• About the point p = 1:

$$z' = X'(1)z = -z \quad \Rightarrow \quad z = Ae^{-t}$$

Given that  $p = p_0$  at  $t = t_0$  and remembering that in this case p = z + 1, we get the linearised solution: o)

$$p = 1 + (p_0 - 1)e^{-(t - t_0)}$$

This is plotted in Fig 2.9 where, in this case, the initial point is taken to be at P. Note that the exact solution and this linear approximation get closer as p approaches 1.



Figure 2.9: Solutions of dp/dt = p(1-p); 3 starting values for  $p(t_0) = p_0$  in the three regions p < 0, 0 and <math>1 < p; two linearisations, one about p = 1 initially passing through P and one about p = 0 initially passing through Q; Exact solution shown in thick line, linear approximations in thin line, asymptotes in dashed line.

# 2.3.2 Classification using linearised equation

An interesting further use of the linearisation about a fixed point is that it can also be employed to classify the corresponding fixed point. Consider again an equation  $\frac{dy}{dx} = X(y)$  and let y = a be a fixed point.

The linearisation equation:

$$\frac{dz}{dx} = zX'(a). \tag{2.9}$$

can be written in terms of the variable y as

$$\frac{dy}{dx} = (y-a)X'(a). \tag{2.10}$$

As we know, classifying the fixed point a means looking at the sign of  $\frac{dy}{dx}$  in the regions y > a and y < a. After linearisation, this sign is determined by the term (y - a)X'(a).

• If X'(a) > 0, then  $\frac{dy}{dx} > 0$  for y > a and  $\frac{dy}{dx} < 0$  for y < a. Therefore the fixed point y = a is a repellor.

- If X'(a) < 0, then  $\frac{dy}{dx} > 0$  for y < a and  $\frac{dy}{dx} < 0$  for y > a. Therefore the fixed point y = a is an attractor.
- If X'(a) = 0 this means that linearisation is not a good enough approximation in this case. The nature of the fixed point can not be determined from the linearised equation.

# Chapter 3

# Solution of Second Order Linear Systems

# 3.1 Introduction

In Chapter 2 we considered a general 1-dimensional autonomous system which we investigated using the method of linearisation. This enabled us to classify the fixed points using just the linearisation, provided the fixed points were repellors or attractors, and additionally find good approximate solutions to the equation close to the fixed points. For the 1-dimensional system there is very little to say, however the methodology can be extended into higher dimensions. We now look at 2-dimensional systems and as before we will start by considering the linear autonomous systems. First however we review the types of 2-dimensional systems we might consider:

# Review of types of 2-dimensional problems

The general format of the 2-dimensional systems we are considering is:

$$\frac{dx_1}{dt} = X_1(x_1, x_2, t) \qquad \frac{dx_2}{dt} = X_2(x_1, x_2, t)$$
(3.1)

and using a vector notation we write:

$$\frac{d\underline{x}}{dt} = \underline{X}(\underline{x}, t)$$

where  $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $\underline{X}(\underline{x}, t) = \begin{pmatrix} X_1(x_1, x_2, t) \\ X_2(x_1, x_2, t) \end{pmatrix}$ .

Example 3.1

$$\frac{dx_1}{dt} = 33 - 10x_1 - 3x_2 + x_1^2 + t = X_1(x_1, x_2, t)$$
  
$$\frac{dx_2}{dt} = -18 + 6x_1 + 2x_2 - x_1x_2 + t^2 = X_2(x_1, x_2, t)$$

# Example 3.2

$$\frac{d^2\theta}{dt^2} = -\sin\theta$$

Setting  $x_1 = \theta$  and  $x_2 = d\theta/dt$  this equation can be rewritten as:

$$\frac{dx_1}{dt} = x_2 = X_1(x_1, x_2, t)$$
$$\frac{dx_2}{dt} = -\sin x_1 = X_2(x_1, x_2, t)$$

## Example 3.3

$$\frac{dx_1}{dt} = 33 - 10x_1 - 3x_2 = X_1(x_1, x_2)$$
$$\frac{dx_2}{dt} = -18 + 6x_1 + 2x_2 = X_2(x_1, x_2)$$

In the above we see that in Example 3.1 t is contained explicitly in the righthand side of both the equations whereas in the other two examples t does not appear. We make the following definition:

#### Definition

The system

$$\frac{d\underline{x}}{dt} = \underline{X}(\underline{x})$$

is said to be **autonomous** since t (the independent variable) is not contained explicitly in the righthand side.

Note:

- Example 3.1 is non-autonomous and non linear. It is said to be non-linear due to the  $x_1^2$  term and the  $x_1x_2$  term, though just one of these terms would have been sufficient to make the system non-linear.
- Example 3.2 is autonomous and non-linear. The non-linearity is due to the  $\sin x_1$  term.
- Example 3.3 is autonomous and linear and can be written in the form

, ,

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} -10 & -3 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 33 \\ -18 \end{pmatrix}$$

We make the following definition:

#### Definition

The system:

$$\frac{d\underline{x}}{dt} = A(t)\underline{x} + \underline{b}(t),$$

where  $\underline{x}$  and  $\underline{b}$  are in  $\mathbb{R}^2$  and A(t) is a 2 × 2 matrix is called an 2<sup>nd</sup> order linear system

If in addition A and  $\underline{b}$  are constant the system is also autonomous. Example 3.3 is an autonomous linear system.

### **Fixed** points

As with the one dimensional system we define the idea of a fixed point of the autonomous equations as follows:

#### Definition

A fixed point of the system:

$$\frac{d\underline{x}}{dt} = \underline{X}(\underline{x}) \tag{3.2}$$

is a constant vector  $\underline{a}$  such that  $\underline{X}(\underline{a}) = \underline{0}$ . We note that  $\underline{x} = \underline{a}$  is also a solution of Eq 3.2.

- In Example 3.2 the fixed points occur at  $x_2 = 0$  and  $-\sin x_1 = 0 \implies x_1 = n\pi$ . Thus the fixed points are on the  $x_1$ -axis at multiples of  $\pi$ .
- In Example 3.3 the fixed points are found by solving:

$$33 - 10x_1 - 3x_2 = 0$$
 and  $-18 + 6x_1 + 2x_2 = 0$ 

which gives  $x_1 = 6$  and  $x_2 = -9$ . Thus this system has only one fixed point at (2, -9).

• We will not consider fixed points of non-autonomous systems.

# **3.2** Linear autonomous second order systems

So far in this chapter we have been looking at systems of two first order linear equations. For those systems using a vector notation as above is useful. It turns out that every second order linear differential equation can be re-written as a system of two first order linear equations as follows: consider the general equation

$$a\frac{d^{2}x}{dt^{2}} + b\frac{dx}{dt} + cx = 0, (3.3)$$

where a, b, c are constants.

If we define  $x = x_1$  and  $\frac{dx}{dt} = x_2$  then Eq 3.3 gives:

$$\frac{dx_1}{dt} = x_2$$
 and  $\frac{dx_2}{dt} = -\frac{c}{a}x_1 - \frac{b}{a}x_2$ 

Which can be written in the standard  $\underline{\dot{x}} = A\underline{x}$  form as:

$$\underline{\dot{x}} = \begin{pmatrix} 0 & 1\\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \begin{pmatrix} x_1\\ x_2 \end{pmatrix}$$
(3.4)

Here  $\underline{\dot{x}} = \frac{dx}{dt}$ .

Thus we are able to obtain a solution of Eq 3.4 by solving Eq 3.3. Rather than do this in general consider the following example

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0.$$

You have learned last year how to solve this kind of equation. We try an exponential type solution  $e^{\lambda t}$ , substitute into the equation and get an equation for  $\lambda$ . In this case the equation is  $\lambda^2 + 3\lambda + 2 = 0$  which gives two solutions  $\lambda_1 = -1$  and  $\lambda_2 = -2$ . Therefore, the general solution to this equation would be  $x = \alpha e^{-t} + \beta e^{-2t}$  where  $\alpha$  and  $\beta$  are arbitrary constants.

Let us now try to solve the same equation  $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$  by rewriting it first in matrix form. As before we define  $x_1 = x$  and  $x_2 = \dot{x}_1$ . We then have equations

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -3x_2 - 2x_1.$$
 (3.5)

In matrix form, the equation can be written as

$$\underline{\dot{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
(3.6)

The new equations (3.5) and their matrix form can now be solved by using properties of the matrix

$$A = \left(\begin{array}{cc} 0 & 1\\ -2 & -3 \end{array}\right).$$

The main properties which will play a crucial role are the eigenvalues and eigenvectors of A. The eigenvalues can be found by computing the determinant

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3 - \lambda \end{vmatrix} = \lambda(3 + \lambda) + 2 = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2).$$
(3.7)

The determinant has zeros at  $\lambda_1 = -1$  and  $\lambda_2 = -2$ . The corresponding eigenvectors associated to these eigenvalues are obtained by solving the equations

$$A\underline{\underline{E}}_1 = -\underline{\underline{E}}_1 \qquad A\underline{\underline{E}}_2 = -2\underline{\underline{E}}_2. \tag{3.8}$$

Explicitly, for the 1st eigenvalue,

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} a \\ b \end{pmatrix},$$
(3.9)

which gives equations b = -a and -2a - 3b = -b. Both equations are in fact equivalent so that any vector with a = -b is an eigenvector of A. We will choose a = -b = 1, although any other choice would also work. This fixes

$$\underline{\underline{E}}_1 = \begin{pmatrix} 1\\ -1 \end{pmatrix}. \tag{3.10}$$

Performing a similar calculation it is possible to show that

$$\underline{E}_2 = \begin{pmatrix} 1\\ -2 \end{pmatrix}. \tag{3.11}$$

You also know from Linear Algebra, that the matrix  $P = (\underline{E}_1, \underline{E}_2)$  whose columns are the eigenvectors above, diagonalizes A. That is,

$$P^{-1}AP = J = \begin{pmatrix} -1 & 0\\ 0 & -2 \end{pmatrix}.$$
 (3.12)

Notice that this also implies  $A = PJP^{-1}$ . All these Linear Algebra results allow us to transform the equation (3.6) into a simpler equation that we can easily solve<sup>1</sup>. If we write (3.6) as

$$\underline{\dot{x}} = A\underline{x} \quad \Leftrightarrow \quad \underline{\dot{x}} = PJP^{-1}\underline{x}. \tag{3.13}$$

Multiplying both sides of the last equation by  $P^{-1}$  we have

$$P^{-1}\underline{\dot{x}} = JP^{-1}\underline{x},\tag{3.14}$$

which means that we just got a new equation  $\underline{y} = J\underline{y}$  for the vector  $\underline{y} = P^{-1}\underline{x}$ . The advantage of this is that the equation for y is very simple to solve. In components, it is

$$\underline{\dot{y}} = J\underline{y} \quad \Leftrightarrow \quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$
(3.15)

Equivalently we have equations  $\dot{y}_1 = -y_1$  and  $\dot{y}_2 = -2y_2$ . These equations can be easily solved because they are independent from each other, that is the equation for  $y_1$  only involves  $y_1$  and the same for  $y_2$ . They have the standard solutions  $y_1 = \alpha e^{-t}$  and  $y_2 = \beta e^{-2t}$ , where  $\alpha$  and  $\beta$  are arbitrary constants.

Once we have y we can get  $\underline{x}$  simply by

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P \underline{y} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha e^{-t} \\ \beta e^{-2t} \end{pmatrix} = \begin{pmatrix} \alpha e^{-t} + \beta e^{-2t} \\ -\alpha e^{-t} - 2\beta e^{-2t} \end{pmatrix}.$$
(3.16)

If we look at the solution for  $x_1 = \alpha e^{-t} + \beta e^{-2t}$  we see that it is exactly the same solution we obtained above for  $x = x_1$  by the usual method. The solution for  $x_2 = -\alpha e^{-t} - 2\beta e^{-2t}$  is nothing but  $\dot{x}_1$  as we expected from our original definition.

We can further look at the structure of (3.16) and draw some general conclusions. We have

$$\underline{x} = \begin{pmatrix} \alpha e^{-t} + \beta e^{-2t} \\ -\alpha e^{-t} - 2\beta e^{-2t} \end{pmatrix} = \alpha e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \alpha e^{\lambda_1 t} \underline{E}_1 + \beta e^{\lambda_2 t} \underline{E}_2.$$
(3.17)

#### General solution

From the example above we can deduce some general results. First, given a linear equation of the

<sup>1</sup>Recall that, given a general  $2 \times 2$  matrix

$$P = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right),$$

its inverse is simply given by

$$P^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

form  $\underline{\dot{x}} = A\underline{x}$ , its general solution is in general a linear combination of fundamental solutions of the form  $e^{\lambda_i t}\underline{E}_i$ , with i = 1, 2 and  $\lambda_i, \underline{E}_i$  being the eigenvalues and eigenvectors associated to the matrix A, respectively. It is quite easy to show that this is true in general.

Let us substitute the solutions  $e^{\lambda_i t} \underline{E}_i$  directly onto the equation  $\underline{\dot{x}} = A\underline{x}$ . We find,

$$\underline{\dot{x}} = A\underline{x} \quad \Rightarrow \quad \frac{de^{\lambda_i t}\underline{E}_i}{dt} = A\left(e^{\lambda_i t}\underline{E}_i\right) \quad \Rightarrow \quad \lambda_i e^{\lambda_i t}\underline{E}_i = e^{\lambda_i t}A\underline{E}_i \quad \Rightarrow \quad \lambda\underline{E}_i = A\underline{E}_i$$

Thus we deduce that if:

$$4\underline{E}_i = \lambda_i \underline{E}_i \tag{3.18}$$

then

 $\underline{x} = e^{\lambda_i t} \underline{E}_i$  is a solution of  $\underline{\dot{x}} = A\underline{x}$ .

It can be shown that if  $\underline{E}_1$  and  $\underline{E}_2$  are two independent solutions of  $A\underline{E} = \lambda \underline{E}$ , with  $\lambda = \lambda_1$  and  $\lambda_2$  respectively, then the general solution of the 2-dimensional system  $\underline{\dot{x}} = A\underline{x}$  is given by:

$$\underline{x} = C_1 e^{\lambda_1 t} \underline{E}_1 + C_2 e^{\lambda_2 t} \underline{E}_2 \tag{3.19}$$

where  $C_1$  and  $C_2$  are arbitrary constants.

Unfortunately it is not always possible to find two vectors,  $\underline{E}_1$  and  $\underline{E}_2$ , that are both independent and real. We therefore need to study the solutions of Eq. (3.18) and their relationship to the solution of  $\underline{\dot{x}} = A\underline{x}$  in more detail.

# 3.3 Linear Algebra - eigenvalues and eigenvectors

We now take time out to review some of the basic results from Linear Algebra that are relevant to our problem. Namely the solutions of Eq. (3.18) and the subsequent idea of reducing A to a Jordan canonical form (in the previous example this would be the matrix J obtained by diagonalising A).

How to work out the eigenvectors and eigenvalues of a matrix should be familiar to you from the Linear Algebra module and I will probably not go through the example below in the class. Questions are welcome in case any difficulties arise.

Predominantly A will be a real constant  $2 \times 2$  matrix, however most of the definitions and results are applicable to the *n*-dimensional case.

### Definition

Given an  $n \times n$  real matrix A then a non-zero vector  $\underline{E}$  such that:

$$A\underline{E} = \lambda \underline{E} \tag{3.20}$$

is called an **eigenvector** corresponding to the **eigenvalue**  $\lambda$ . Equation 3.20 is called the **eigenvalue** equation for A.

Note:

- In this definition  $\underline{E} \neq \underline{0}$  however  $\lambda$  may be zero.
- From equation 3.19 we see how the eigenvalues and vectors may feature in our solution.
### 3.3.1 Evaluation of eigenvalues and eigenvectors

Consider the following example:

Given 
$$A = \begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix}$$
 find  $\underline{E} = \begin{pmatrix} a \\ b \end{pmatrix} \neq \underline{0}$  and  $\lambda$  such that  $A \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$ 

Thus we are required to find a, b (not both zero) and  $\lambda$  such that:

$$\left(\begin{array}{cc} 4 & -6\\ 3 & -5 \end{array}\right) \left(\begin{array}{c} a\\ b \end{array}\right) = \lambda \left(\begin{array}{c} a\\ b \end{array}\right)$$

In terms of simultaneous equations this gives:

$$4a - 6b = \lambda a \implies (4 - \lambda)a - 6b = 0$$
$$3a - 5b = \lambda b \implies 3a + (-5 - \lambda)b = 0$$

Writing this in matrix form we have;

$$\left(\begin{array}{cc} 4-\lambda & -6\\ 3 & -5-\lambda \end{array}\right) \left(\begin{array}{c} a\\ b \end{array}\right) = \underline{0}$$

or even more concisely:

$$(A - \lambda I) \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \tag{3.21}$$

where I is the usual  $2 \times 2$  identity matrix. To obtain a, b and  $\lambda$  we argue as follows:

- If  $(A \lambda I)$  has an inverse then  $\begin{pmatrix} a \\ b \end{pmatrix} = (A \lambda I)^{-1} \underline{0} = \underline{0}$  Thus the only solution is a = b = 0 which is not permissible for an eigenvector.
- Thus we require  $\lambda$  such that  $(A \lambda I)$  has no inverse. The condition that a matrix has no inverse is that its determinant is zero. Thus we require  $\lambda$  such that

$$\det(A - \lambda I) = 0 \tag{3.22}$$

Equation 3.22 is called the characteristic equation for A.

### General result

The eigenvalues of A are given by the roots of the characteristic equation:

$$\det(A - \lambda I) = 0$$

In general when A is  $n \times n$  this is an  $n^{th}$  degree polynomial equation. In our example, as we see below, it is a quadratic.

### To find the eigenvalues

Returning to our problem:

$$\det \begin{pmatrix} 4-\lambda & -6\\ 3 & -5-\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad (4-\lambda)(-5-\lambda) - (-6)(3) = 0 \quad \Rightarrow$$

 $\lambda^2 + \lambda - 2 = 0 \quad \Rightarrow \quad \lambda_1 = +1 \text{ and } \lambda_2 = -2$ 

labelling the two values of  $\lambda$  as  $\lambda_1$  and  $\lambda_2$ .

### To find the eigenvectors

For each value of  $\lambda$  we need to find a and b (not both zero) such that:

$$\left(\begin{array}{cc} 4-\lambda & -6\\ 3 & -5-\lambda \end{array}\right) \left(\begin{array}{c} a\\ b \end{array}\right) = \underline{0}$$

 $\lambda_1 = 1$ 

$$\begin{pmatrix} 4-1 & -6 \\ 3 & -5-1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow$$
$$\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad \begin{array}{c} 3a-6b = & 0 \\ 3a-6b = & 0 \end{array} \quad \Rightarrow \quad a = 2b$$

Thus corresponding to  $\lambda_1 = 1$  we have

$$\left(\begin{array}{c}a\\b\end{array}\right) = \left(\begin{array}{c}2b\\b\end{array}\right) = b\left(\begin{array}{c}2\\1\end{array}\right)$$

 $\lambda_2 = -2$ 

$$\begin{pmatrix} 4+2 & -6\\ 3 & -5+2 \end{pmatrix} \begin{pmatrix} a\\ b \end{pmatrix} = \underline{0} \quad \Rightarrow$$
$$\begin{pmatrix} 6 & -6\\ 3 & -3 \end{pmatrix} \begin{pmatrix} a\\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad \begin{array}{c} 6a-6b = & 0\\ 3a-3b = & 0 \end{array} \quad \Rightarrow \quad a = b$$

Thus corresponding to  $\lambda_2 = -2$  we have

$$\left(\begin{array}{c}a\\b\end{array}\right) = \left(\begin{array}{c}a\\a\end{array}\right) = a\left(\begin{array}{c}1\\1\end{array}\right).$$

In each case we see that we do not obtain a unique eigenvector but linear multiple of a vector. If we draw a line through the origin along the direction of an eigenvector, this is called the eigenline corresponding to the eigenvector. In the above we set

$$\underline{E}_1 = \begin{pmatrix} 2\\1 \end{pmatrix}$$
 and  $\underline{E}_2 = \begin{pmatrix} 1\\1 \end{pmatrix}$ 

These eigenvectors alone define two eigenlines (the lines that go by the origin and the points (2, 1) and (1, 1), respectively). These eigenlines will play an important role later on when we draw the phase diagrams associated to systems of equations like the one of section 3.2.

### 3.3.2 Canonical Forms

Let us think again about the example of section 3.2. A crucial step in our solution procedure was the fact that we could diagonalise the matrix A by multiplying it by the matrix  $P = (\underline{E}_1, \underline{E}_2)$ . When doing this we got a diagonal matrix  $J = P^{-1}AP$  in terms of which the system of equations simplified considerably.

A very special feature of the matrix A in section 3.2 is that it had two different, real eigenvalues  $\lambda_1 = -1, \lambda_2 = -2$ . Because of this, it also had two linearly independent eigenvectors in terms of which we could construct the matrix P.

In order to be able to solve any systems of two linear equations we need to answer the question: how can we solve such a system if the eigenvalues are equal to each other or non real? How can we construct the matrices P and J in such a case?

The general result goes as follows:

### Theorem

Given a  $2 \times 2$  real matrix A there exists a non-singular real matrix P such that:

$$A = PJP^{-1}$$

where J is one of the four matrices:

- If the eigenvalues of A are  $\lambda_1$  and  $\lambda_2$  with  $\lambda_1 \neq \lambda_2$  and  $\lambda_1, \lambda_2 \in \mathbb{R}$  then  $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ .
- If the eigenvalues of A are  $\lambda_1 = \lambda_2$  with  $\lambda_1 \in \mathbb{R}$  then  $J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ .
- If the eigenvalues of A are  $\lambda_1 = \alpha + i\beta$  and  $\lambda_2 = \alpha i\beta$  with  $\alpha, \beta \in \mathbb{R}$  and  $\beta > 0$  then  $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ .
- If  $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$  with  $\lambda_1$  real, then J = A.

### J is called the canonical or Jordan Normal form of A.

We will now analyse these four cases in more detail. We need to answer two questions: first, how to find the matrix P that relates A to J in each of the four cases above and second, once we have found P and J, how to solve the resulting system of equations.

### **Case 1** $\lambda_1$ and $\lambda_2$ real and different

As we saw in the example discussed earlier, the matrix P which relates A to its Jordan Normal form J in this case is simply

$$P = (\underline{E}_1, \underline{E}_2), \quad \text{with} \quad \underline{E}_1, \underline{E}_2 \quad \text{eigenvectors of } A, \quad (3.23)$$

and

$$J = \begin{pmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{pmatrix}. \tag{3.24}$$

If we want to solve the equation  $\underline{\dot{x}} = A\underline{x}$  we just have to follow the following steps:

$$\underline{\dot{x}} = A\underline{x} \quad \Leftrightarrow \quad \underline{\dot{x}} = PJP^{-1}\underline{x} \quad \Leftrightarrow \quad P^{-1}\underline{\dot{x}} = JP^{-1}\underline{x}. \tag{3.25}$$

Defining now the vector  $\underline{y} = P^{-1}\underline{x}$ , we see that the original equation  $\underline{\dot{x}} = A\underline{x}$  has been transformed into the new equation

$$\underline{\dot{y}} = J\underline{y},\tag{3.26}$$

which is the Canonical form of the original equation. This equation does not look too different, but it is much easier to solve because J is the diagonal matrix (3.24). We have,

$$\underline{\dot{y}} = J\underline{y} \quad \Leftrightarrow \quad \left(\begin{array}{c} \dot{y}_1\\ \dot{y}_2 \end{array}\right) = \left(\begin{array}{c} \lambda_1 & 0\\ 0 & \lambda_2 \end{array}\right) \left(\begin{array}{c} y_1\\ y_2 \end{array}\right) \quad \Leftrightarrow \quad \dot{y}_1 = \lambda_1 y_1, \, \dot{y}_2 = \lambda_2 y_2, \tag{3.27}$$

therefore

$$y_1 = C_1 e^{\lambda_1 t}$$
 and  $y_2 = C_2 e^{\lambda_2 t}$ , with  $C_1, C_2$  arbitrary constants. (3.28)

This can also be written as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Having now solved the Canonical form of the system of equation we still need to find the solution for  $\underline{x}$ . From before we have that

$$\underline{x} = P\underline{y} = (\underline{E}_1, \underline{E}_2)\underline{y} = C_1 e^{\lambda_1 t} P\begin{pmatrix} 1\\0 \end{pmatrix} + C_2 e^{\lambda_2 t} P\begin{pmatrix} 0\\1 \end{pmatrix} = C_1 e^{\lambda_1 t} \underline{E}_1 + C_2 e^{\lambda_2 t} \underline{E}_2.$$
(3.29)

**Example:** Here is a concrete example of a system belonging to case 1. As an exercise work out the solutions below.

Consider the system of equations

$$\dot{x}_1 = 7x_1 - 2x_2, \qquad \dot{x}_2 = 2x_1 + 2x_2.$$

• In matrix form this is  $\underline{\dot{x}} = A\underline{x}$  with

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 7 & -2 \\ 2 & 2 \end{pmatrix}.$$

• The matrix A has the following eigenvalues and eigenvectors (check this!):

$$\lambda_1 = 6, \quad \lambda_2 = 3 \quad \text{and} \quad \underline{E}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \underline{E}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Notice that if you multiply these eigenvectors by a real number you will still get an eigenvector. Any choice is allowed!

• The matrix P is given by

$$P = \left(\begin{array}{cc} 2 & 1\\ 1 & 2 \end{array}\right),$$

and therefore, its inverse is

$$P^{-1} = \frac{1}{3} \left( \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right).$$

You can check that

$$P^{-1}AP = J = \left(\begin{array}{cc} 6 & 0\\ 0 & 3 \end{array}\right).$$

• The vector y which solves the system of equations in its Canonical form is

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{6t} \\ C_2 e^{3t} \end{pmatrix} = C_1 e^{6t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which gives the final solution

$$\underline{x} = P\underline{y} = C_1 e^{6t} P \begin{pmatrix} 1\\0 \end{pmatrix} + C_2 e^{3t} P \begin{pmatrix} 0\\1 \end{pmatrix} = C_1 e^{6t} \underline{E}_1 + C_2 e^{3t} \underline{E}_2 = C_1 e^{6t} \begin{pmatrix} 2\\1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1\\2 \end{pmatrix},$$

or, in components,

$$x_1 = 2C_1e^{6t} + C_2e^{3t} \qquad x_2 = C_1e^{6t} + 2C_2e^{3t}$$

**Case 2**  $\lambda_1 = \lambda_2$ , one real eigenvalue.

When both eigenvalues are equal it is not possible to find two independent eigenvectors as in the previous case. This means that in order to construct the matrix P we need the eigenvector  $\underline{E}_1$  associated to the single eigenvalue  $\lambda_1$  and one other vector which we will call  $\underline{J}_1$ . The matrix P is then constructed as before

$$P = (\underline{E}_1, \underline{J}_1), \tag{3.30}$$

where  $\underline{J}_1$  is a vector which can be constructed as follows: Linear algebra guarantees that in this case there will exist a vector  $\underline{J}_1$  such that

$$(A - \lambda_1 I)\underline{J}_1 = \underline{E}_1. \tag{3.31}$$

Therefore, once we have computed the eigenvector  $\underline{E}_1$ , we just have to substitute in (3.31) and solve for the vector  $\underline{J}_1$ . Once this has been done, the solution procedure is very similar to that for case 1. The equations (3.25) and (3.26) hold as before with P given by (3.30) and J given by

$$J = \left(\begin{array}{cc} \lambda_1 & 1\\ 0 & \lambda_1 \end{array}\right).$$

This means that the equation for the vector y in this case is

$$\underline{\dot{y}} = J\underline{y} \qquad \Leftrightarrow \qquad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

or, in components,

$$\dot{y}_1 = \lambda_1 y_1 + y_2, \qquad \dot{y}_2 = \lambda_1 y_2.$$

The second equation is of the same type as for case 1 and is solved by  $y_2 = C_2 e^{\lambda_1 t}$ , where  $C_2$  is an arbitrary constant. Substituting into the first equation we have,

$$\dot{y}_1 = \lambda_1 y_1 + C_2 e^{\lambda_1 t},$$

which we can rewrite as

$$\dot{y}_1 - \lambda_1 y_1 = C_2 e^{\lambda_1 t}$$

This is the typical equation that is solved by using an integrating factor. We multiply the equation by a function R(t),

$$R(t)\dot{y}_1 - R(t)\lambda_1 y_1 = R(t)C_2 e^{\lambda_1 t}.$$

If

$$R(t)\dot{y}_1 - R(t)\lambda_1 y_1 = \frac{d(R(t)y_1)}{dt},$$

then

$$R(t) = e^{-\int \lambda_1 dt} = e^{-\lambda_1 t + C} = A e^{-\lambda_1 t},$$

with C an arbitrary constant and  $A = e^{C}$ . Once we have fixed R(t) we can write the equation as

$$\frac{d(R(t)y_1)}{dt} = R(t)C_2e^{\lambda_1 t} \qquad \Leftrightarrow \qquad \frac{d(Ae^{-\lambda_1 t}y_1))}{dt} = Ae^{-\lambda_1 t}C_2e^{\lambda_1 t} = AC_2,$$

therefore

$$e^{-\lambda_1 t} y_1 = C_2 \int dt \quad \Leftrightarrow \quad y_1 = e^{\lambda_1 t} (C_2 t + C_1),$$

where  $C_1$  is an arbitrary constant. Therefore the final solutions are

 $y_1 = (C_1 + tC_2)e^{\lambda_1 t}$  and  $y_2 = C_2 e^{\lambda_1 t}$ .

To obtain the solutions in the original coordinates we have to compute

$$\underline{x} = P\underline{y} = (C_1 + tC_2)e^{\lambda_1 t}\underline{E}_1 + C_2 e^{\lambda_1 t}\underline{J}_1.$$

Example: Consider

$$A = \left(\begin{array}{cc} 0 & 4\\ -1 & 4 \end{array}\right)$$

We calculate the eigenvalues and vectors in the usual way,

- $|A \lambda I| = \begin{vmatrix} 0 \lambda & 4 \\ -1 & 4 \lambda \end{vmatrix} = -\lambda(4 \lambda) + 4 = \lambda^2 4\lambda + 4 = (\lambda 2)^2 = 0$ Thus we have a single eigenvalue  $\lambda_1 = 2$ .
- Set  $\underline{E}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$  hence  $A\underline{E}_1 = 2\underline{E}_1 \implies \begin{pmatrix} 0-2 & 4 \\ -1 & 4-2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$ Thus  $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2b \\ b \end{pmatrix} = b \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and we take  $\underline{E}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- We now introduce a new vector to form the second axis. Linear algebra guarantees that in this case there will exist a vector  $\underline{J}_1$  such that

$$(A - \lambda_1 I)\underline{J}_1 = \underline{E}_1 \tag{3.32}$$

where I is the identity matrix. To calculate  $\underline{J}_1$  we substitute  $\underline{J}_1 = \begin{pmatrix} m \\ n \end{pmatrix}$  into Eq. (3.32) to give:

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \implies m = 2n - 1$$

Thus

$$\left(\begin{array}{c}m\\n\end{array}\right) = \left(\begin{array}{c}2n-1\\n\end{array}\right) = n\left(\begin{array}{c}2\\1\end{array}\right) + \left(\begin{array}{c}-1\\0\end{array}\right)$$

Again we have an infinite choice of vector, thus we make the simplest choice with n = 0 and take  $\underline{J}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ . Any other value of n would have been equally valid.

• The matrix P is given by:

$$P = (\underline{E}_1, \underline{J}_1) = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$$

and it relates the matrix A above to its Jordan form

$$J = \left(\begin{array}{cc} 2 & 1\\ 0 & 2 \end{array}\right) = PAP^{-1}.$$

Following the general construction above, the solution for the vector y is

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (C_1 + C_2 t)e^{2t} \\ C_2 e^{2t} \end{pmatrix} = (C_1 + C_2 t)e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and therefore, the solutions for  $\underline{x} = P\underline{y}$  are

$$\underline{x} = (C_1 + C_2 t)e^{2t}P\begin{pmatrix}1\\0\end{pmatrix} + C_2 e^{2t}P\begin{pmatrix}0\\1\end{pmatrix} = (C_1 + C_2 t)e^{2t}\underline{E}_1 + C_2 e^{2t}\underline{J}_1$$

or, in components,

$$x_1 = 2(C_1 + C_2 t)e^{2t} - C_2 e^{2t}$$
 and  $x_2 = (C_1 + C_2 t)e^{2t}$ .

**Case 3**  $\lambda_1$  and  $\lambda_2$  complex

The eigenvalues of A are complex if when solving the characteristic polynomial the roots of the quadratic are complex.

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$$

Without loss of generality we will always assume that  $\beta > 0$ .

In general, for the  $n \times n$  case, provided A is real any complex roots of the characteristic equation will appear in conjugate pairs, as above.

Furthermore since A is real:

If 
$$A\underline{E}_1 = \lambda_1\underline{E}_1 \implies \overline{A\underline{E}_1} = \overline{\lambda_1\underline{E}_1} \implies A(\underline{\bar{E}}_1) = \overline{\bar{\lambda}}_1\underline{\bar{E}}_1$$

Where the *over bar* represents the complex conjugate. Thus we see that not only do the eigenvalues occur in conjugate pairs but so do the corresponding eigenvectors.

ie if  $(\lambda_1, \underline{E}_1)$  is an eigenvalue/vector pair then so is  $(\overline{\lambda}_1, \underline{\overline{E}}_1)$ .

As in case 1, we could use the eigenvectors  $\underline{E}_1$  and  $\underline{E}_2$  to construct the matrix P that transforms A to its Jordan Normal form. However, the eigenvectors  $\underline{E}_1$  and  $\underline{E}_2$  are now complex, which would lead to a complex matrix P. We are interested however in working always with real quantities. Therefore to avoid this problem we can instead choose a different pair of vectors, which is real to construct our matrix P. The vectors we will use are not  $\underline{E}_1$  and  $\underline{E}_2$  but the real and imaginary parts of  $\underline{E}_1$  instead. It turns out that the real and imaginary parts of  $\underline{E}_1$  are real vectors and furthermore it can be shown that they are independent.

We define,

 $\underline{e}_1 = \operatorname{Re}(\underline{E}_1)$  and  $\underline{e}_2 = \operatorname{Im}(\underline{E}_1)$ 

 $\underline{e}_1$  and  $\underline{e}_2$  are used now to construct the matrix

$$P = (\underline{e}_1, \underline{e}_2)$$

which transforms A to its Jordan Normal form

$$J = PAP^{-1} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \tag{3.33}$$

and performs the change of coordinates by the usual formula  $\underline{x} = Py$ .

We now have to use these results to solve the equation  $\underline{\dot{x}} = A\underline{x}$ , when A has complex conjugated eigenvalues. In its canonical form the equation becomes,

$$\underline{\dot{y}} = J\underline{y} \qquad \Leftrightarrow \qquad \left(\begin{array}{c} \dot{y}_1\\ \dot{y}_2 \end{array}\right) = \left(\begin{array}{c} \alpha & \beta\\ -\beta & \alpha \end{array}\right) \left(\begin{array}{c} y_1\\ y_2 \end{array}\right) \qquad \Rightarrow \qquad \begin{array}{c} \dot{y}_1 = & \alpha y_1 + \beta y_2\\ \dot{y}_2 = & -\beta y_1 + \alpha y_2 \end{array}$$

To solve these equations we transform to polar coordinates using the substitutions  $y_1 = r \cos \theta$  and  $y_2 = r \sin \theta$  with  $r \ge 0$  to obtain:

$$\dot{y}_1 = \dot{r}\cos\theta - r\theta\sin\theta = \alpha r\cos\theta + \beta r\sin\theta$$
(*i*)  

$$\dot{y}_2 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta = -\beta r\cos\theta + \alpha r\sin\theta$$
(*i*)

To eliminate  $\dot{\theta}$  we form  $(i) \times \cos \theta + (ii) \times \sin \theta$  to give:

$$\dot{r} = \alpha r \quad \Rightarrow \quad \underline{r = r_0 e^{\alpha t}}$$
 where  $r_0$  is an arbitrary constant

Similarly  $(i) \times \sin \theta - (ii) \times \cos \theta$  gives:

 $-r\dot{\theta} = \beta r \quad \Rightarrow \quad \dot{\theta} = -\beta \quad \Rightarrow \quad \underline{\theta} = -\beta t + \theta_0$  where  $\theta_0$  is an arbitrary constant

Thus

$$\underline{y_1 = r_0 e^{\alpha t} \cos(-\beta t + \theta_0)} \qquad \underline{y_2 = r_0 e^{\alpha t} \sin(-\beta t + \theta_0)}, \tag{3.34}$$

or, in vector form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = r_0 e^{\alpha t} \cos(-\beta t + \theta_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{\alpha t} \sin(-\beta t + \theta_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus

$$\underline{x} = P\underline{y} = (\underline{e}_1, \underline{e}_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1\underline{e}_1 + y_2\underline{e}_2 = r_0e^{\alpha t}\cos(-\beta t + \theta_0)P\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0e^{\alpha t}\sin(-\beta t + \theta_0)P\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

 $= r_0 e^{\alpha t} \left\{ \cos(-\beta t + \theta_0)\underline{e}_1 + \sin(-\beta t + \theta_0)\underline{e}_2 \right\}$ 

**Example:** Given  $A = \begin{pmatrix} 6 & -13 \\ 2 & -4 \end{pmatrix}$  find the eigenvalues of A and hence write down its canonical form. Find the eigenvectors of A and hence construct a non-singular matrix P such that Eq. (3.33) holds.

• We first find the eigenvalues in the usual way

$$|A - \lambda I| = \begin{vmatrix} 6 - \lambda & -13 \\ 2 & -4 - \lambda \end{vmatrix} = (6 - \lambda)(-4 - \lambda) + 26 = \lambda^2 - 2\lambda + 2 = 0$$

Solving this quadratic gives  $\lambda_1 = 1 + i$  and  $\lambda_2 = 1 - i$ , hence  $\alpha = 1$  and  $\beta = 1$  and the canonical form is given by:

$$\left(\begin{array}{ccc} 1 & 1 \\ -1 & 1 \end{array}\right)$$

• With  $\lambda_1 = 1 + i$  we now compute the corresponding eigenvector. Let  $\underline{E}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$  the eigenvector is given by:

$$\begin{pmatrix} 6-(1+i) & -13\\ 2 & -4-(1+i) \end{pmatrix} \begin{pmatrix} a\\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad \begin{array}{c} (5-i)a-13b & = & 0\\ 2a-(5+i)b & = & 0 \end{array}$$
 (i) (ii)

The two equation can be shown to be dependent with  $(5 + i) \times (i) = 13 \times (ii)$ , thus we can use either (i) or (ii) to form our eigenvector.

From (*ii*) 
$$\frac{a}{b} = \frac{5+i}{2} \Rightarrow \underline{E}_1 = \begin{pmatrix} 5+i\\ 2 \end{pmatrix}$$

where we have made the most straightforward choice of a and b to give a value of  $\underline{E}_1$ . As usual any non-zero multiple of this  $\underline{E}_1$  would have been just as good. The other eigenvector would just be the complex conjugate of  $\underline{E}_1$ , that is

$$\underline{E}_2 = \overline{\underline{E}}_1 = \begin{pmatrix} 5-i\\2 \end{pmatrix},$$

and we do not need to compute it.

• In order to construct the matrix P we have to obtain the two vectors given by the real and imaginary parts of  $\underline{E}_1$ , that is

$$\underline{e}_1 = \operatorname{Re}\begin{pmatrix}5+i\\2\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$$
 and  $\underline{e}_2 = \operatorname{Im}\begin{pmatrix}5+i\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ 

Thus  $P = \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix}$  and by direct calculation you can verify that :

$$A = \begin{pmatrix} 6 & -13 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix}^{-1}$$

• So far we have found,

$$A = \begin{pmatrix} 6 & -13 \\ 2 & -4 \end{pmatrix}, \quad \alpha = 1, \quad \beta = 1, \quad \underline{e}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \underline{e}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

therefore, following the general steps above, the solution of  $\underline{\dot{x}} = A\underline{x}$  is given by

$$\underline{x} = r_0 e^t \left\{ \cos(-t + \theta_0) \begin{pmatrix} 5\\2 \end{pmatrix} + \sin(-t + \theta_0) \begin{pmatrix} 1\\0 \end{pmatrix} \right\}$$

In terms of components this breaks down into:

$$x_1 = r_0 e^t \{ 5\cos(-t + \theta_0) + \sin(-t + \theta_0) \}$$
 and  $x_2 = 2r_0 e^t \cos(-t + \theta_0)$ 

#### Case 4 $A = \lambda_1 I$

This is a trivial case since if A is a multiple of the identity then the equation  $\underline{\dot{x}} = A\underline{x}$  can be solved directly and the matrix A can be considered to be already in canonical form. The equation to be solved is

$$\underline{\dot{x}} = A\underline{x} \quad \Leftrightarrow \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which in components becomes

$$\dot{x}_1 = \lambda_1 x_1, \qquad \dot{x}_2 = \lambda_1 x_2,$$
 $a_1 = C_1 e^{\lambda_1 t}$  and  $x_2 = C_2 e^{\lambda_1 t}$ 

$$(2.25)$$

which are solved by

$$x_1 = C_1 e^{\lambda_1 t}$$
 and  $x_2 = C_2 e^{\lambda_1 t}$ . (3.35)

# Chapter 4

# Phase Diagrams for Second Order Linear Systems

# 4.1 Simple fixed points

This section deals with the classification of fixed points of the linear system  $\underline{\dot{x}} = A\underline{x} + \underline{b}$  and the construction of the phase diagrams about the fixed points. For most of the section, we will be concentrating on the case when the vector  $\underline{b} = 0$  (as we did in the previous chapter).

Recall that for one-dimensional systems (or systems of one equation) there were only three types of fixed points: attractors, repellors and shunts. For systems of two equations there are many more possibilities, as the phase diagram is two dimensional. There are in fact *ten* different types of fixed points which we will be classifying and studying in this chapter!

The fixed points of the system are given by the solutions of the equation  $A\underline{x} + \underline{b} = \underline{0}$ . If  $A^{-1}$  exists, then there exists a unique fixed point given by  $\underline{x} = -A^{-1}\underline{b}$ . Thus, if  $\underline{b} = 0$  the fixed point is at the origin.

We will restrict our attention to the case where  $A^{-1}$  exists, is A is non-singular. We already know that  $A^{-1}$  exists if and only if  $\det(A) \neq 0$ , however we now show that this is equivalent to A not having a zero eigenvalue.

Consider the characteristic polynomial det $(A - \lambda I)$  and assume that  $\lambda_1$  and  $\lambda_2$  are the eigenvalues of A. ( $\lambda_1$  and  $\lambda_2$  may be real and different, equal or complex) Thus we can write:

$$det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Since this equality is true for all  $\lambda$ , setting  $\lambda = 0$  gives the result that  $\det(A) = \lambda_1 \lambda_2$ . Thus  $\det(A) \neq 0$  if and only if neither  $\lambda_1$  nor  $\lambda_2$  are equal to zero. Since we are only going to consider the case when A is non-singular we make the following definition:

### Definition

 $\underline{x} = \underline{a}$  is a simple fixed point of  $\underline{\dot{x}} = A\underline{x} + \underline{b}$  if:

- it is a solution of the equation  $A\underline{a} + \underline{b} = 0$
- A is non-singular. Equivalently this is  $det(A) \neq 0$ ,  $A^{-1}$  exists, no eigenvalue of A is zero.

The fixed point is unique and given by  $\underline{a} = -A^{-1}\underline{b}$ 

In the case  $\underline{b} = 0$ , that is to say if the equation is  $\underline{\dot{x}} = A\underline{x}$ , the only simple fixed point is at the origin.

Initially we restrict our attention to the problem  $\underline{\dot{x}} = A\underline{x}$  and then later look at the more general problem with  $\underline{b} \neq 0$ .

# 4.2 Classification of Simple Fixed Points

Before developing the phase diagrams we first write down the classifications of the simple fixed point of the system  $\underline{\dot{x}} = A\underline{x} + \underline{b}$ . These are listed according to the eigenvalues of A.

- 1. Eigenvalues real and different
  - saddle  $\lambda_1$  and  $\lambda_2$  different signs: take  $\lambda_1 > 0 > \lambda_2$
  - **unstable node**  $\lambda_1$  and  $\lambda_2$  both positive: take  $\lambda_1 > \lambda_2 > 0$
  - stable node  $\lambda_1$  and  $\lambda_2$  both negative: take  $0 > \lambda_2 > \lambda_1$
- 2. Single eigenvalue  $(\lambda_1 = \lambda_2)$ 
  - unstable improper node  $\lambda_1 > 0$
  - stable improper node  $\lambda_1 < 0$
- 3. Complex eigenvalues  $\alpha \pm i\beta$  with  $\beta > 0$ 
  - unstable focus  $\alpha > 0$
  - stable focus  $\alpha < 0$
  - centre  $\alpha = 0$
- 4.  $A = \lambda_1 I$ 
  - unstable star node  $\lambda_1 > 0$
  - stable star node  $\lambda_1 < 0$

# 4.3 Phase Diagram Type I: $\lambda_1 \neq \lambda_2$ , real and non-zero

In order to obtain the phase diagram for  $\underline{\dot{x}} = A\underline{x}$  we first obtain the phase diagram for the canonical system  $\dot{y} = Jy$ . In this case the canonical form is given by

$$\underline{\dot{y}} = \begin{pmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{pmatrix} \underline{y} \quad \Rightarrow \quad y_1 = C_1 e^{\lambda_1 t} \quad y_2 = C_2 e^{\lambda_2 t} \quad \Rightarrow \quad y_2 = C y_1^{\lambda_2/\lambda_1} \tag{4.1}$$

The phase diagram, although given by this expression, will depend on the signs of  $\lambda_1$  and  $\lambda_2$ . Thus we consider the three possible scenarios, namely where the  $\lambda$ 's are both positive, both negative or of different sign.

### 4.3.1 $\lambda_1 > 0 > \lambda_2$ : saddle

From Eq 4.1 we can deduce that with these values of lambda:

- $\lambda_2 < 0 \quad \Rightarrow \quad y_2 \to 0 \quad \text{as} \quad t \to \infty$
- $\lambda_2 < 0 \quad \Rightarrow \quad y_2 \to \infty \quad \text{as} \quad t \to -\infty$
- $\lambda_1 > 0 \quad \Rightarrow \quad y_1 \to 0 \quad \text{as} \quad t \to -\infty$
- $\lambda_1 > 0 \quad \Rightarrow \quad y_1 \to \infty \quad \text{as} \quad t \to \infty$
- If  $C_1 = 0$  then the trajectory is  $y_1 = 0$ , that is the  $y_2$ -axis is part of the phase diagram. In this case  $y_2 = C_2 e^{\lambda_2 t}$  which means that  $|y_2|$  grows as as t tends to minus infinity and viceversa (this determines the direction of the arrows in the phase diagram).
- If  $C_2 = 0$  then the trajectory is  $y_2 = 0$ , that is the  $y_1$ -axis is also part of the phase diagram, as we can see below. In this case  $y_1 = C_1 e^{\lambda_1 t}$ , that is to say  $|y_1|$  tending to infinity as t tends to infinity.

The phase diagram for the canonical case is seen in Fig. 4.1(a). To find the phase diagram for the full system  $\underline{\dot{x}} = A\underline{x}$  we recall that  $\underline{x} = Py$  where P is the matrix whose columns are the eigenvectors of A.

If  $\underline{E}_1$  and  $\underline{E}_2$  are the eigenvectors of A corresponding to the eigenvalues  $\lambda_1$  and  $\lambda_2$  respectively then

$$\left(\begin{array}{c} x_1\\ x_2 \end{array}\right) = \left(\begin{array}{c} \underline{E}_1 & \underline{E}_2 \end{array}\right) \left(\begin{array}{c} y_1\\ y_2 \end{array}\right)$$

Thus we see that the unit vector  $\begin{pmatrix} 1\\0 \end{pmatrix}$  along the  $y_1$ -axis is transformed to  $\underline{E}_1$  in the  $x_1$ - $x_2$  plane. That is to say the  $y_1$  axis is transformed into the line of  $\underline{E}_1$ . Similarly the  $y_2$ -axis is transformed into the line of  $\underline{E}_2$ . Thus we see that the diagram is deformed, squashing or stretching the trajectories. Under



Figure 4.1: (a) canonical saddle with  $\lambda_1 = 1$  and  $\lambda_2 = -2$ . (b) the transformed saddle to give the phase diagram for  $\underline{\dot{x}} = A\underline{x}$ 

this transformation the direction of the arrows on the  $y_1$ - $y_2$  axes is maintained along the lines of  $\underline{E}_1$ 

and  $\underline{E}_2$ . This is a useful way of determining the direction of the arrows for the whole diagram. The diagram has been drawn using the example of Section 3.3.1 where  $\lambda_1 = 1$ ,  $\lambda_2 = -2$ ,  $\underline{E}_1 = \begin{pmatrix} 2\\1 \end{pmatrix}$  and  $\underline{E}_2 = \begin{pmatrix} 1\\1 \end{pmatrix}$ . In Fig.4.1(b) the heavy lines are the lines of the eigenvectors  $\underline{E}_1$  and  $\underline{E}_2$  respectively.

4.3.2  $\lambda_1 > \lambda_2 > 0$ : unstable node

From Eq. 4.1 we can deduce

$$\frac{dy_2}{dy_1} = \frac{\frac{dy_2}{dt}}{\frac{dy_1}{dt}} = \frac{\lambda_2 C_2 e^{\lambda_2 t}}{\lambda_1 C_1 e^{\lambda_1 t}} = \frac{\lambda_2}{\lambda_1} C e^{-(\lambda_1 - \lambda_2)t}$$

where  $C_1$  and  $C_2$  have been combined to give C. Hence:

• As  $t \to \infty$   $\frac{dy_2}{dy_1} \to 0$  thus the trajectories become horizontal.

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- As  $t \to -\infty$   $|\frac{dy_2}{dy_1}| \to \infty$  thus the trajectories become vertical.
- Both  $y_1$  and  $y_2$  tend to zero as t tends to minus infinity. (from Eq. 4.1)
- Both  $|y_1|$  and  $|y_2|$  tend to infinity at t tends to plus infinity. (from Eq. 4.1)

Additionally from Eq. 4.1 we can see that both the axes are trajectories with arrows pointing away from the origin as follows:

- If  $C_1 = 0$  then the trajectory is  $y_1 = 0$  and  $y_2 = C_2 e^{\lambda_2 t}$ , that is to say the  $y_2$ -axis with  $|y_2|$  tending to zero as t tends to infinity.
- If  $C_2 = 0$  then the trajectory is  $y_2 = 0$  and  $y_1 = C_1 e^{\lambda_1 t}$ , that is to say the  $y_1$ -axis with  $|y_1|$  tending to infinity as t tends to infinity.

Fig. 4.2 is drawn using the data from example 4.1 where the eigenvalues are +6 and +3 and the eigenvectors are  $\underline{E}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $\underline{E}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ . The lines along these direction are shown in fig. 4.2(b) in bold and are respectively the transformations of the  $y_1$  and  $y_2$  axes.

### Example 4.1

Given

$$\dot{x} = 7x_1 - 2x_2 \qquad \dot{x}_2 = 2x_1 + 2x_2$$

- Locate and classify the fixed point.
- Sketch the phase diagram.
- Obtain the solution that passes through the point (1,1) at t = 0.



Figure 4.2: (a) canonical unstable node with  $\lambda_1 = 6$  and  $\lambda_2 = 3$ . (b) the transformed unstable node to give the phase diagram for  $\underline{\dot{x}} = A\underline{x}$ 

This is in fact the same example we did in page 37, so we already know the fixed points, eigenvectors and eigenvalues. We will take the solutions we found there and concentrate largely on drawing the phase diagram. Recall the solutions:

$$\underline{x} = C_1 e^{\lambda_1 t} \underline{E}_1 + C_2 e^{\lambda_2 t} \underline{E}_2 = C_1 e^{6t} \begin{pmatrix} 2\\1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1\\2 \end{pmatrix}$$

Or in terms of components:

$$x_1 = 2C_1e^{6t} + C_2e^{3t} \qquad x_2 = C_1e^{6t} + 2C_2e^{3t}$$

We are given the initial condition that at t = 0,  $x_1 = x_2 = 1$ , which allows us to fix the constants as

$$1 = 2C_1 + C_2$$
 and  $1 = C_1 + 2C_2 \Rightarrow C_1 = C_2 = 1/3$ 

Thus the solutions are:

$$x_1 = \frac{1}{3} \left( 2e^{6t} + e^{3t} \right)$$
  $x_2 = \frac{1}{3} \left( e^{6t} + 2e^{3t} \right)$ 

The phase diagram is drawn in Fig. 4.2(b) The important features are that the trajectories become parallel to the image of the  $y_1$ -axis, that is to say parallel to  $\underline{E}_1$  as  $t \to \infty$  and appear to emanate from the origin parallel to the image of the  $y_2$ -axis, that is to say parallel to  $\underline{E}_2$ .

# 4.3.3 $\lambda_1 < \lambda_2 < 0$ : stable node

Provided we arrange for  $|\lambda_1| > |\lambda_2|$ , that is to say  $\lambda_1$  is now the most negative eigenvalue rather than the most positive, then the phase diagram for the stable node is exactly the same as the one for the unstable node but with the arrows reversed.

# 4.4 Phase Diagram Type II: $\lambda_1 = \lambda_2 \neq 0$

This is the case where the quadratic characteristic equation has only a single or repeated root which leads to a single independent eigenvector. This type does not include the case where  $A = \lambda I$  since in this case any vector is an eigenvector and the problem is virtually trivial; see Type IV. As in Type I. we consider the canonical case first with  $\lambda_1 > 0$  and then transform to obtain the phase diagram for  $\underline{\dot{x}} = A\underline{x}$ .

# 4.4.1 $\lambda_1 = \lambda_2 > 0$ : unstable improper node

The canonical form is given by

$$\underline{\dot{y}} = \begin{pmatrix} \lambda_1 & 1\\ 0 & \lambda_1 \end{pmatrix} \underline{y} \quad \Rightarrow \quad y_1 = e^{\lambda_1 t} (C_1 t + C_2) \quad y_2 = C_1 e^{\lambda_1 t}$$
(4.2)

The most useful thing we can do is to calculate the gradient of the phase paths in terms of, initially t and then in terms of  $y_1$  and  $y_2$  as follows:

• Differentiating  $y_1$  and  $y_2$  from Eq. (4.2) with respect to t gives

$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{C_1\lambda_1}{C_1 + \lambda_1C_1t + \lambda_1C_2}$$

Thus as  $t \to \infty$ ,  $\frac{dy_2}{dy_1} \to 0$  and the trajectories become horizontal. Similarly as  $t \to -\infty$ ,  $\frac{dy_2}{dy_1} \to 0$  and again the trajectories are horizontal. This is in contrast to the case of the unstable node where the trajectories became vertical as  $t \to -\infty$ 

• Directly from the canonical equations in (4.2)  $\dot{y}_1 = \lambda_1 y_1 + y_2$  and  $\dot{y}_2 = \lambda_1 y_2$  thus:

$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{\lambda_1 y_2}{\lambda_1 y_1 + y_2}$$

Hence the trajectories are vertical along the line  $y_2 = -\lambda_1 y_1$ . Note that in the unstable case this line has negative gradient since  $\lambda_1 > 0$  and is shown in dots in Fig 4.3(a). The line is useful when sketching the diagram however transforming it into the  $x_1$ - $x_2$  phase space does not give a line on which the trajectories are still vertical. To locate such a line in the  $x_1$ - $x_2$  phase space it is necessary to refer back to the original problem and draw the line  $\dot{x}_1 = 0$ .

• Finally we note from Eq 4.2 that as  $t \to -\infty$  the trajectories tend to the origin, since both  $y_1$  and  $y_2$  tend to zero, and as  $t \to \infty$  both  $|y_1|$  and  $|y_2|$  tend to infinity.

### Example 4.2

Fig. 4.3 is obtained using the following example:

Given

$$\underline{\dot{x}} = \left(\begin{array}{cc} 1 & 1\\ -1 & 3 \end{array}\right) \underline{x}$$

obtain the phase diagram and the general solution.



Figure 4.3: (a) canonical unstable improper node with  $\lambda_1 = 2$ ; the line of vertical trajectories is given by  $2y_1 + y_2 = 0$ . (b) the transformed diagram. The line of vertical trajectories now obtained from  $\dot{x} = 0$  which gives  $x_1 + x_2 = 0$  (see example)

• First calculating the eigenvalues we obtain:

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 1 \\ -1 & 3 - \lambda \end{vmatrix} = (1 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 2$$

•  $\underline{\lambda} = 2$  The eigenvector is given by:

$$\begin{pmatrix} 1-2 & 1 \\ -1 & 3-2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad a = b \quad \Rightarrow \quad \underline{E_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

• The  $\underline{J}_1$  vector, which is required for the solution but not for the diagram is given by:

$$(A - \lambda I)\underline{J_1} = \underline{E_1} \quad \Rightarrow \quad \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad a = b - 1$$
  
Thus  $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b - 1 \\ b \end{pmatrix} = b \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ 

We can take any value of b to give a  $\underline{J_1}$ , thus we take the simplest value, namely b = 0 to give  $\underline{J_1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ .

• The phase diagram is shown in Fig 4.3(b) where we note that the trajectories leave the origin with increasing t since the node is unstable. Additionally as the node is improper the trajectories are parallel to  $\underline{E_1}$  both as they leave the origin and as they tend to infinity. ie they tend to become parallel to  $\underline{E_1}$  as  $|t| \to \infty$ . To help sketch the diagram we have plotted in dotted format the line along which the trajectories are vertical. This is given by setting  $\dot{x}_1 = 0$  which in this example gives  $x_1 + x_2 = 0$ .

• Using the vectors  $\underline{E}_1$  and  $\underline{J}_1$  and the eigenvalue  $\lambda = 2$  the solution is given by

$$\underline{x} = e^{2t}(C_1t + C_2) \begin{pmatrix} 1\\1 \end{pmatrix} + C_1 e^{2t} \begin{pmatrix} -1\\0 \end{pmatrix}$$

or in terms of components:

$$x_1 = e^{2t}(C_1t + C_2 - C_1)$$
 and  $x_2 = e^{2t}(C_1t + C_2)$ 

### **4.4.2** $\lambda_1 = \lambda_2 < 0$ : stable improper node

Unfortunately a simple reversal of arrows is not sufficient to describe the difference between the unstable and stable case. If  $\lambda_1 < 0$  then the gradient of the dotted line along which the trajectories are vertical in the canonical diagram is now positive. In practice for the improper node it is best to work with the final picture and draw in the line along which  $\dot{x}_1 = 0$ , draw in the eigenvector  $\underline{E}_1$  and add the arrows according to whether the node is stable (*towards the origin*) or unstable (*away from the origin*).

# 4.5 Phase Diagram Type Ill: $\lambda_1$ and $\lambda_2$ complex $\alpha \pm i\beta$

The canonical equation for this type is given by

$$\underline{\dot{y}} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \underline{y} \qquad \beta > 0$$

As we saw in Eq.3.34 the solution is given by

$$y_1 = r\cos\theta = r_0 e^{\alpha t}\cos(-\beta t + \theta_0) \quad \text{and} \quad y_2 = r\sin\theta = r_0 e^{\alpha t}\sin(-\beta t + \theta_0) \tag{4.3}$$

A simple way to see the shape of the phase space trajectories in this case is to notice that

$$y_1^2 + y_2^2 = r_0^2 e^{2\alpha t}$$

- If  $\alpha = 0$ , this is the equation of a circle of radius  $r_0$ . The phase space trajectories are concentric circles, centered at the origin (see later in section 4.5.3).
- If  $\alpha > 0$ , then the equation is not that of a circle anymore, but we can think of it as a "circle" whose radius increases exponentially with time. In the phase diagram the trajectories will look like curves that spiral out of the origin and go to infinity as  $t \to \infty$ .
- If  $\alpha < 0$ , then the equation is not that of a circle either, but we can think of it as a "circle" whose radius decreases exponentially with time. In the phase diagram the trajectories will look like curves that spiral in towards the origin and go to infinity as  $t \to -\infty$ .

Although these equations represent simple curves in the canonical picture the transformation to the  $x_1$ - $x_2$  plane will distort and may change the orientation of the solutions. As we see below, it is essential that we track any change in orientation and try to figure out how the trajectories may get distorted in the original coordinates.

### **4.5.1** $\alpha > 0$ : unstable focus

From the polar coordinates it is clear that with positive  $\alpha$ , as t increases, r, the distance from the origin, increases exponentially, whereas  $\theta$  decreases linearly with time. This mean that as t increase a point P will move away from the origin and rotate clockwise about the origin. This is illustrated in Fig 4.4(a). Looking at the transformation matrix to transform the canonical picture to the  $x_1$ - $x_2$  plane is not very fruitful and the only thing we can usefully do is to check the direction of the spiral in the  $x_1$ - $x_2$  plane by looking at the sign of  $\dot{x}_1$  when  $x_1 = 0$ , that is to say where the trajectories cut the  $x_2$ -axis. This is illustrated in Fig 4.4(b) which is constructed from the example below.



Figure 4.4: (a) Canonical unstable focus with  $\lambda_1 = 1 + i$  and  $\lambda_2 = 1 - i$  (b) the transformed diagram.

### Example 4.3

Fig. 4.4 is obtained using the following example:

Given

$$\underline{\dot{x}} = \left(\begin{array}{cc} -2 & 2\\ -5 & 4 \end{array}\right) \underline{x}$$

obtain the phase diagram and the general solution.

• Eigenvalues are found by solving the characteristic equation:

$$\begin{vmatrix} -2-\lambda & 2\\ -5 & 4-\lambda \end{vmatrix} = (-2-\lambda)(4-\lambda) + 10 = \lambda^2 - 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = 1+i \quad \lambda_2 = 1-i$$

Thus the fixed point is an unstable focus.

• The eigenvector is given by

$$\begin{pmatrix} -2-1-i & 2\\ -5 & 4-1-i \end{pmatrix} \begin{pmatrix} a\\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad (-3-i)a+2b = 0$$
$$\Rightarrow \quad \underline{E_1} = \begin{pmatrix} 2\\ 3+i \end{pmatrix} = \begin{pmatrix} 2\\ 3 \end{pmatrix} + \begin{pmatrix} 0\\ 1 \end{pmatrix} i = \underline{e_1} + \underline{e_2}i$$

• The solution is given by:

$$\underline{x} = r_0 e^t \left( \cos(-t + \theta_0) \underline{e_1} + \sin(-t + \theta_0) \underline{e_2} \right) = r_0 e^t \left( \cos(-t + \theta_0) \begin{pmatrix} 2\\3 \end{pmatrix} + \sin(-t + \theta_0) \begin{pmatrix} 0\\1 \end{pmatrix} \right)$$

In components this gives:

$$x_1 = 2r_0 e^t \cos(-t + \theta_0) \qquad x_2 = r_0 e^t \left(3\cos(-t + \theta_0) + \sin(-t + \theta_0)\right)$$

• The phase diagram consists of spirals moving away from the origin since  $\alpha = 1 > 0$ . To check the direction of the spirals we observe the direction of the trajectories as they cross the  $x_2$ -axis by considering the sign of  $\dot{x}_1$  at  $x_1 = 0$ . In this example  $\dot{x}_1 = -2x_1 + 2x_2$  thus on the  $x_2$ -axis,  $x_1 = 0$  and  $\dot{x}_1 = 2x_2$ , which is positive if  $x_2 > 0$ . Thus the trajectories crossing the positive  $x_2$ -axis move in the direction of increasing  $x_1$  and hence are moving in a clockwise direction. See Fig. 4.4(b).

### **4.5.2** $\alpha < 0$ : stable focus

In the canonical picture the only change is that the spiral now tend towards the origin rather than away, however the diagram will need to be redrawn since the rotation is still clockwise. In the  $x_1$ - $x_2$ plane the trajectories still spiral into the origin however as before it will be necessary to check the orientation. (*clockwise or anti-clockwise*)



Figure 4.5: (a) Canonical centre (b) Centre for the system  $\underline{\dot{x}} = A\underline{x}$  (see example below)

### **4.5.3** $\alpha = 0$ : centre

With  $\alpha = 0$  it is clear from the solution to the canonical problem (Eq 4.3) that  $r = r_0$ , thus the trajectories are concentric circles. In the canonical case the orientation is clockwise however in the  $x_1$ - $x_2$  plane the orientation may be different and will have to be checked as before using the sign of  $\dot{x}_1$ . The following examples is a centre and is illustrated in Fig 4.5(b).

$$\underline{\dot{x}} = \left(\begin{array}{cc} -128 & -272\\ 80 & 128 \end{array}\right) \underline{x}$$

In the one dimensional case we mentioned the idea of stability and asymptotic stability (See section 2.2.) This is an example of a fixed point that is stable but not asymptotically stable. All the other stable fixed points considered so far in this chapter have also been asymptotically stable as they tend to the fixed point as t tend to infinity.

# **4.6** Phase Diagram Type IV: $A = \lambda_1 I$

This is the most trivial of the cases and really only arises after linearising a nonlinear problem (see Chapter 5). From Eq. (3.35) the solution is given by  $x_1 = C_1 e^{\lambda_1 t}$  and  $x_2 = C_2 e^{\lambda_1 t}$ . Thus dividing these two expressions and relabelling the constants we get that the trajectories are given by  $x_2 = Cx_1$ , with  $C = C_2/C_1$ . Thus the trajectories are straight lines through the origin pointing away from the origin if  $\lambda_1 > 0$  to give an **unstable star node** and pointing into the origin if  $\lambda_1 < 0$  to give a **stable star node**.



Figure 4.6: Unstable star node;  $\underline{\dot{x}} = 2I\underline{x}$ 

# 4.7 General Linear System

We conclude this chapter with the solution and phase diagram for the complete linear system  $\underline{\dot{x}} = A\underline{x} + \underline{b}$  (so far, we have been looking only at the case  $\underline{b} = 0$ ). The fixed points of this system are given by the solution of the equation  $A\underline{a} + \underline{b} = 0$ , however since we are only considering simple fixed points, that is to say A is nonsingular, the fixed point is unique and given by  $\underline{a} = -A^{-1}\underline{b}$ . If we move the origin to the fixed point using the substitution  $\underline{x} = \underline{a} + \underline{z}$  then:

$$\underline{\dot{x}} = A\underline{x} + \underline{b} \quad \Rightarrow \quad \underline{\dot{x}} = \underline{\dot{z}} = A(\underline{a} + \underline{z}) + \underline{b} = A\underline{a} + A\underline{z} + \underline{b}$$

Since  $\underline{a}$  is a fixed point  $A\underline{a} + \underline{b} = 0$  thus  $\underline{\dot{z}} = A\underline{z}$ .

Thus the solution and phase diagram for  $\underline{\dot{x}} = A\underline{x} + \underline{b}$  is obtained as follows:

• Solve  $A\underline{x} + \underline{b} = 0$  to give the fixed point  $\underline{a}$ 

- Solve  $\underline{\dot{z}} = A\underline{z}$  for  $\underline{z}$
- Write  $\underline{x} = \underline{a} + \underline{z}$ .
- From Fig. 4.7 we can see that the substitution  $\underline{x} = \underline{a} + \underline{z}$  represents a change of origin, thus the phase diagram for  $\underline{\dot{x}} = A\underline{x} + \underline{b}$  is obtained by drawing the phase diagram for  $\underline{\dot{z}} = A\underline{z}$  about the new origin.



Figure 4.7: Change of origin from O to O '

### Example 4.4

Obtain the solution to :

 $\dot{x}_1 = 4x_1 - 6x_2 - 6$   $\dot{x}_2 = 3x_1 - 5x_2 - 4$ 

such that  $x_1 = 3$  and  $x_2 = 4$  at t = 0.

Draw the phase diagram for the system.

• Writing the system in the form  $\underline{\dot{x}} = A\underline{x} + \underline{b}$  gives:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -6 \\ -4 \end{pmatrix}$$

• The fixed point for the system is given by:

$$\underline{a} = \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix}^{-1} \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -5 & 6 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Alternatively you can solve the simultaneous equations  $4x_1 - 6x_2 - 6 = 0$  and  $3x_1 - 5x_2 - 4 = 0$ by direct elimination. • The eigenvalues and eigenvectors for this matrix A have already been evaluated in Sec. 3.3.1 to give  $\lambda_1 = 1$  and  $\lambda_2 = -2$  with  $\underline{E_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $\underline{E_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . Thus the solution of  $\underline{\dot{z}} = A\underline{z}$  is given by:

$$\underline{z} = C_1 e^t \begin{pmatrix} 2\\1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1\\1 \end{pmatrix}$$

and the fixed point at  $\underline{z} = 0$  is a saddle about this point.

Substituting for  $\underline{x} = \underline{a} + \underline{z}$  gives:

$$\underline{x} = \begin{pmatrix} 3\\1 \end{pmatrix} + C_1 e^t \begin{pmatrix} 2\\1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1\\1 \end{pmatrix}$$

Applying the initial conditions that  $x_1 = 3$  and  $x_2 = 4$  at t = 0 gives:

$$\begin{pmatrix} 3\\4 \end{pmatrix} = \begin{pmatrix} 3\\1 \end{pmatrix} + C_1 \begin{pmatrix} 2\\1 \end{pmatrix} + C_2 \begin{pmatrix} 1\\1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0\\3 \end{pmatrix} = C_1 \begin{pmatrix} 2\\1 \end{pmatrix} + C_2 \begin{pmatrix} 1\\1 \end{pmatrix}$$

Thus we have two equations for  $C_1$  and  $C_2$ ,  $2C_1 + C_2 = 0$  and  $C_1 + C_2 = 3$ . Solving gives  $C_1 = -3$  and  $C_2 = 6$ , hence the particular trajectory through the point (3, 4) is given by:

$$\underline{x} = \begin{pmatrix} 3\\1 \end{pmatrix} - 3e^t \begin{pmatrix} 2\\1 \end{pmatrix} + 6e^{-2t} \begin{pmatrix} 1\\1 \end{pmatrix}$$

In terms of components this gives the two solutions:

$$x_1 = 3 - 6e^t + 6e^{-2t} \qquad x_2 = 1 - 3e^t + 6e^{-2t}$$



Figure 4.8: Saddle about the fixed point (3, 1) for  $\underline{\dot{x}} = A\underline{x} + \underline{b}$ 

• The phase diagram is given in Fig 4.8. Note that this is a saddle drawn about the fixed point at (3,1) as origin. The eigenvectors  $\underline{E_1}$  and  $\underline{E_2}$  are drawn relative to this origin and thus the trajectories move out along the line of  $\underline{E_1}$  and in towards this origin along the line of  $\underline{E_2}$ .

# Chapter 5

# Nonlinear Second Order Systems

In this chapter we study two dimensional autonomous nonlinear systems of differential equations. In general such systems are not solvable in terms of elementary functions such as exponentials, polynomials and trigonometric functions. The method of approach is to linearise the systems about their fixed points and then construct the phase diagrams accordingly to give overall or global pictures. The linear solutions can also be used to approximate the exact solutions near the fixed points. This approach was seen in the one dimensional case in Section 2.3.

# 5.1 Linearisation in 2-dimensions

Let  $\underline{x} = \underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$  be a fixed point of the system  $\underline{\dot{x}} = \underline{X}(\underline{x})$ . (ie.  $\underline{X}(\underline{a}) = \underline{0}$ )

Move the origin to the fixed point using the substitution  $\underline{x} = \underline{z} + \underline{a}$ . (See fig 4.7)

Thus  $\underline{\dot{x}} = \underline{\dot{z}} = \underline{X}(\underline{z} + \underline{a})$  or expressing this in coordinates:

$$\dot{z}_1 = X_1(z_1 + a_1, z_2 + a_2)$$
 and  $\dot{z}_2 = X_2(z_1 + a_1, z_2 + a_2)$ 

We now expand the righthand sides of these equations using Taylor's expansion in two variables to give:

$$\dot{z}_{1} = X_{1}(a_{1}, a_{2}) + z_{1} \frac{\partial X_{1}(a_{1}, a_{2})}{\partial x_{1}} + z_{2} \frac{\partial X_{1}(a_{1}, a_{2})}{\partial x_{2}} + \frac{1}{2!} \left( z_{1}^{2} \frac{\partial^{2} X_{1}(a_{1}, a_{2})}{\partial x_{1}^{2}} + 2z_{1} z_{2} \frac{\partial^{2} X_{1}(a_{1}, a_{2})}{\partial x_{1} \partial x_{2}} + z_{2}^{2} \frac{\partial^{2} X_{1}(a_{1}, a_{2})}{\partial x_{2}^{2}} \right) + \dots$$
(5.1)

$$\dot{z}_{2} = X_{2}(a_{1}, a_{2}) + z_{1} \frac{\partial X_{2}(a_{1}, a_{2})}{\partial x_{1}} + z_{2} \frac{\partial X_{2}(a_{1}, a_{2})}{\partial x_{2}} + \frac{1}{2!} \left( z_{1}^{2} \frac{\partial^{2} X_{2}(a_{1}, a_{2})}{\partial x_{1}^{2}} + 2z_{1} z_{2} \frac{\partial^{2} X_{2}(a_{1}, a_{2})}{\partial x_{1} \partial x_{2}} + z_{2}^{2} \frac{\partial^{2} X_{2}(a_{1}, a_{2})}{\partial x_{2}^{2}} \right) + \dots$$
(5.2)

Since  $\underline{x} = \underline{a}$  is a fixed point we have that  $\underline{X}_1(a_1, a_2)$  and  $\underline{X}_2(a_1, a_2)$  are both zero. If additionally we stay close to  $\underline{x} = \underline{a}$  both  $z_1$  and  $z_2$  are small and thus in Eq. (5.1) and Eq. (5.2) we can ignore all

terms of second order and above (that is to say  $z_1^2$ ,  $z_2^2$ ,  $z_1z_2$  and above). Hence we form the linear approximation as follows:

$$\dot{z}_1 = z_1 \frac{\partial X_1(a_1, a_2)}{\partial x_1} + z_2 \frac{\partial X_1(a_1, a_2)}{\partial x_2}$$
(5.3)

$$\dot{z}_2 = z_1 \frac{\partial X_2(a_1, a_2)}{\partial x_1} + z_2 \frac{\partial X_2(a_1, a_2)}{\partial x_2}$$
(5.4)

This can be written in matrix form as:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix}_{(a_1, a_2)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$
(5.5)

The notation  $(a_1, a_2)$  in Eq 5.5 indicates that the partial derivatives in the matrix should be evaluated at  $x_1 = a_1$  and  $x_2 = a_2$ .

The matrix in Eq 5.5 expressed at the general point  $(x_1, x_2)$  is called the **Jacobian** matrix of  $\underline{X}(\underline{x})$ .

We now consider an example of a nonlinear system and attempt to construct its global phase diagram by initially constructing the phase diagrams close to its fixed points using the linearised equations about the fixed points.

### Example 5.1

Consider the following equations:

$$\dot{x}_1 = x_1 + x_2^2 = X_1$$
 and  $\dot{x}_2 = x_1^2 + x_2 = X_2$ 

### 1. Find the fixed points

The fixed points occur when  $X_1(x_1, x_2) = X_2(x_1, x_2) = 0$ . Thus

$$x_1 + x_2^2 = 0$$
 (i) and  $x_1^2 + x_2 = 0$  (ii)

Substituting for  $x_1$  from (i) into (ii) gives  $x_2^4 + x_2 = 0$  which implies that  $x_2 = 0$  or  $x_2 = -1$ . Substituting these values back into (i) gives respectively  $x_1 = 0$  and  $x_1 = -1$ .

Thus we have two fixed points, one at (0,0) and one at (-1,-1).

### 2. Obtain the Jacobian matrix

This is given by:

$$A = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 2x_2 \\ x_1 & 1 \end{pmatrix}$$
(5.6)

### 3. Linearisation at (0,0).

Using Eq 5.6 we obtain the linearisation about (0,0) to be:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2x_2 \\ 2x_1 & 1 \end{pmatrix}_{(0,0)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Thus the linearisation is an **unstable star node** at the origin. The assumption will be that the nonlinear system behaves very much like this close to the origin.

### 4. Linearisation at (-1, -1).

Using Eq 5.6 we obtain the linearisation about (-1, 1) to be:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2x_2 \\ 2x_1 & 1 \end{pmatrix}_{(-1,-1)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

To classify this fixed point and draw its phase diagram we proceed as usual:

• The eigenvalues are given by:

$$\begin{vmatrix} 1-\lambda & -2\\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$

Thus  $\lambda_1 = 3$  and  $\lambda_2 = -1$ . The fixed point for the linearisation is therefore a **saddle point**. Again the assumption will be that the nonlinear system is similar to the linear system in the neighbourhood of the fixed point.

• The eigenvectors are given by:  $\lambda_1 = 3$ 

$$\begin{pmatrix} 1-3 & -2 \\ -2 & 1-3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

Thus we see that -2a - 2b = 0 which gives  $\underline{E}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ 

 $\lambda_2 = -1$ 

$$\begin{pmatrix} 1+1 & -2 \\ -2 & 1+1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} +2 & -2 \\ -2 & +2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

Thus we see that +2a - 2b = 0 which gives  $\underline{E}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 

### 5. Local phase diagram

The phase diagrams for these two fixed points are shown in Fig 5.1. Note how the saddle is placed at (-1, -1) and the directions  $\underline{E_1}$  and  $\underline{E_2}$  are shown relative to a set of axes (*denoted in dots*) through this point. To obtain the global phase diagram we now have to join these two diagrams together in a consistent manner. This is not always that easy and in some cases more than one construction is possible, though of course only one is correct.

The second phase diagram in Fig 5.1 is the true phase diagram for the nonlinear system, we can see how the saddle and star node obtained by linearising about the two fixed points are linked



Figure 5.1: Hand drawn linear phase diagrams about the two fixed points; Nonlinear global picture

to give the global picture.

In this rather simple nonlinear problem it is possible to see that the line  $x_2 = x_1$  is a trajectory in the global picture. (shown as a heavy line in the global picture in Fig 5.1)

To show that  $x_2 = x_1$  is a trajectory we show that it is a solution of the first order differential equation defining the phase diagram as follows:

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{x_1^2 + x_2}{x_1 + x_2^2} \tag{5.7}$$

Substituting  $x_2 = x_1$  into each side of this equation gives:

$$\frac{dx_2}{dx_1} = \frac{dx_1}{dx_1} = 1$$
 and  $\frac{x_1^2 + x_2}{x_1 + x_2^2} = \frac{x_1^2 + x_1}{x_1 + x_1^2} = 1$ 

Thus  $x_2 = x_1$  is a solution of Eq 5.7 and hence is a trajectory in the phase plane.

Finally if we require an approximation to the solution of our nonlinear system near the fixed point (-1, -1) then we can use the solution of the linearised problem about this point. In this case this is given by:

$$\underline{x} = \underline{a} + \underline{z} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + C_1 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or in component form:

$$x_1 = -1 - C_1 e^{3t} + C_2 e^{-t}$$
 and  $x_2 = -1 + C_1 e^{3t} + C_2 e^{-t}$ 

To illustrate how the phase diagram might change with only a small change in the original system consider the following example:

## Example 5.2

Consider the following equations:

$$\dot{x}_1 = x_1^2 + x_2$$
 and  $\dot{x}_2 = x_1 + x_2^2$ 

### 1. Find the fixed points

The fixed points occur when:

$$x_1^2 + x_2 = 0$$
 (i) and  $x_1 + x_2^2 = 0$  (ii)

Substituting for  $x_2$  from (i) into (ii) gives  $x_1^4 + x_1 = 0$  which implies that  $x_1 = 0$  or  $x_1 = -1$ . Substituting these values back into (i) gives respectively  $x_2 = 0$  and  $x_2 = -1$ .

Thus the two fixed points are still at (0,0) and (-1,-1).

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### 2. Obtain the Jacobian matrix

This is given by:

$$A = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 1 \\ 1 & 2x_2 \end{pmatrix}$$
(5.8)

### 3. Linearisation at (0,0).

Using Eq 5.8 we obtain the linearisation about (0,0) to be:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 2x_1 & 1 \\ 1 & 2x_2 \end{pmatrix}_{(0,0)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

To classify this fixed point:

• The eigenvalues are given by:

$$\begin{vmatrix} 0 - \lambda & 1 \\ 1 & 0 - \lambda \end{vmatrix} = (-\lambda)^2 - 1 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0$$

Thus  $\lambda_1 = 1$  and  $\lambda_2 = -1$ . The fixed point for the linearisation is therefore a saddle.

• The eigenvectors are given by:  $\lambda_1 = 1$ 

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

Thus we see that -a + b = 0 which gives  $\underline{E_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 

 $\lambda_2 = -1$ 

 $\left(\begin{array}{rr}1&1\\1&1\end{array}\right)\left(\begin{array}{r}a\\b\end{array}\right)=\underline{0}$ Thus we see that a + b = 0 which gives  $\underline{E}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ 

4. Linearisation at (-1, -1).

Using Eq. (5.8) we obtain the linearisation about (-1, 1) to be:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 2x_1 & 1 \\ 1 & 2x_2 \end{pmatrix}_{(-1,-1)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

To classify this fixed point:

• The eigenvalues are given by:

$$\begin{vmatrix} -2 - \lambda & 1 \\ 1 & -2 - \lambda \end{vmatrix} = (-2 - \lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1) = 0$$

Thus  $\lambda_1 = -3$  and  $\lambda_2 = -1$ . The fixed point for the linearisation is therefore a **stable** node.

• The eigenvectors are given by:  $\lambda_1 = -3$ 

$$\begin{pmatrix} -2+3 & 1\\ 1 & -2+3 \end{pmatrix} \begin{pmatrix} a\\ b \end{pmatrix} = \begin{pmatrix} 1 & 1\\ 1 & 1 \end{pmatrix} \begin{pmatrix} a\\ b \end{pmatrix} = \underline{0}$$

Thus we see that a + b = 0 which gives  $\underline{E_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ 

 $\underline{\lambda_2 = -1}$ 

$$\begin{pmatrix} -2+1 & 1\\ 1 & -2+1 \end{pmatrix} \begin{pmatrix} a\\ b \end{pmatrix} = \begin{pmatrix} -1 & 1\\ 1 & -1 \end{pmatrix} \begin{pmatrix} a\\ b \end{pmatrix} = \underline{0}$$

Thus we see that -a + b = 0 which gives  $\underline{E}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 

### 5. Local phase diagram

The phase diagrams for these two fixed points are shown in Fig 5.2 Note how the stable node is placed at (-1, -1) and the directions  $\underline{E_1}$  and  $\underline{E_2}$  are shown relative to a set of axes (*denoted* in dots) through this point. The saddle is placed at (0,0) and the directions  $\underline{E_1}$  and  $\underline{E_2}$  for this point are again indicated. To obtain the global phase diagram we now have to join these two diagrams together in a consistent manner.

The second phase diagram in Fig 5.2 is the true phase diagram for the nonlinear system, we can see how the saddle and stable node obtained by linearising about the two fixed points are linked to give the global picture.

As in the last problem it is possible to see that the line  $x_2 = x_1$  is a trajectory in the global picture. (shown in heavy line in the global picture in Fig 5.2)

Finally if we require an approximation to the solution of our nonlinear system near the fixed point (-1, -1) then we can again use the solution of the linearised problem about this point. In this case this is given by:

$$\underline{x} = \underline{a} + \underline{z} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + C_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



Figure 5.2: Hand drawn linear phase diagrams about the two fixed points; Nonlinear global picture

or in component form:

$$x_1 = -1 - C_1 e^{-3t} + C_2 e^{-t}$$
 and  $x_2 = -1 + C_1 e^{-3t} + C_2 e^{-t}$ 

The above examples bear out what we would expect, that is the nonlinear problem and its linearisation about its fixed points have very similar phase diagrams and solutions in the neighbourhood of the fixed points. However the following example shows that this is not always the case.

### Example 5.3

Given the following differential equations analyse the phase diagram and solution in the neighbourhood of the origin:

$$\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2)$$
  $\dot{x}_2 = x_1 - x_2(x_1^2 + x_2^2)$  (5.9)

The equations clearly have a fixed point at the origin and thus to investigate the phase diagram near the origin we linearise the equations about  $x_1 = x_2 = 0$ . The Jacobian matrix is given by:

$$A = \begin{pmatrix} -3x_1^2 - x_2^2 & -1 - 2x_1x_2 \\ 1 - 2x_1x_2 & -x_1^2 - 3x_2^2 \end{pmatrix}$$

Thus at the origin the linearisation is:

$$\underline{\dot{z}} = \left(\begin{array}{cc} 0 & -1\\ 1 & 0 \end{array}\right) \underline{z}$$

Evaluating the eigenvalues of A we obtain:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = +i \qquad \lambda_2 = -i$$

which indicates that the origin is a centre, or at least for the linearised problem it is. However a centre is a very delicate state for a system, it only needs a small change for the real part of the complex eigenvalue to become non-zero. In such a case the centre will turn into a focus. This example has been chosen so that we can solve it exactly and see precisely what is happening at the origin; and elsewhere for that matter. If we change to polar coordinates and substitute  $x_1 = r \cos \theta$  and  $x_2 = r \sin \theta$  into Eq 5.9 we obtain:

$$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = -r\sin\theta - r^3\cos\theta \quad (i) \qquad \dot{r}\sin\theta + r\dot{\theta}\cos\theta = r\cos\theta - r^3\sin\theta \quad (ii)$$

Which gives, eliminating in turn  $\dot{r}$  and  $\dot{\theta}$  from the left hand sides of these equation:

$$(i) \times \cos \theta + (ii) \times \sin \theta \Rightarrow \dot{r} = -r^3 \text{ and } (i) \times \sin \theta - (ii) \times \cos \theta \Rightarrow \dot{\theta} = 1$$

The equation for  $\theta$ , namely,  $\dot{\theta} = 1$ , gives  $\theta = t + \theta_0$ , where as usual  $\theta_0$  is the value of  $\theta$  at t = 0. This implies that as time evolves any point on a trajectory will move steadily anticlockwise about the origin. The question is does it move away from the origin, towards the origin or perform a closed path around the origin as in the case of a centre? Turning our attention to the equation for r, namely  $\dot{r} = -r^3$  we get:

$$\dot{r} = -r^3 \quad \Rightarrow \quad \int -\frac{1}{r^3} dr = \int dt \quad \Rightarrow \quad \frac{1}{2r^2} = t + C \quad \Rightarrow \quad r = \frac{1}{\sqrt{2(t+C)}}$$
(5.10)

where the sign for the square root has been taken to be positive, since r, the distance from the origin is always positive or zero. From Eq 5.10 we see that as t tends to infinity r tends to zero. Thus all the trajectories will tend to and rotate anticlockwise about the the origin as t increases. Thus the origin is in truth more like a stable focus for the nonlinear problem and not a centre as indicated by linearisation. The linearised and nonlinear phase diagrams are shown in Fig 5.3. The



Figure 5.3: Linearised centre; Nonlinear stable focus - slow convergence to the origin

linear problem is just a canonical centre but in the anticlockwise direction whereas the original nonlinear problem looks more like a stable focus at the origin, again moving in the anticlockwise direction.

As t increases the convergence to the origin of the nonlinear trajectories is slower than that found for the stable canonical focus. The reason for this is that for the canonical focus convergence to the origin is exponential, due to the  $e^{\alpha t}$  term in the solution. In this example we see from the solution of the nonlinear equation (Eq 5.10) that r clearly tends to zero as t increases, but only does so like  $1/\sqrt{t}$ 

The above example clearly showed that a centre in the linearised problem could in reality be a stable focus in the nonlinear problem. It is also true that if the fixed point is a centre in the linearised problem it may be an unstable focus in the original nonlinear problem. Finally we must consider the question is it possible for the nonlinear and linearised versions to both be centres at a fixed point? The following example for the sine pendulum, which we saw in Chapter 1, is an example that has such fixed points, where linearising does not change the nature of the fixed points and centres in the linear problem correspond to centres in the nonlinear problem. Strictly speaking we have not defined the idea of a centre for a nonlinear system, however we will take it to mean that in every neighbourhood of the fixed point there exists closed trajectories surrounding the fixed point.

### Example 5.4 - Sine pendulum

The equation of the sine pendulum considered in Chapter 1 is given by  $\ddot{\theta} = -\sin\theta$  and writing this in standard form gives:  $\dot{x}_1 = x_2$  and  $\dot{x}_2 = -\sin x_1$ , with  $x_1 = \theta$  and  $x_2 = \dot{\theta}$ .

The fixed points are on the  $x_1$  axis  $(x_2 = 0)$  where  $\sin x_1 = 0$ , which implies that  $x_1 = n\pi$ . To linearise we form the Jacobian matrix:

$$\begin{pmatrix} 0 & 1 \\ -\cos x_1 & 0 \end{pmatrix}$$
  
Which equals  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  at  $x_1 = 2k\pi$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  at  $x_1 = (2k-1)\pi$ 

At even multiple of  $\pi$  (ie where  $x_1 = 2k\pi$ ) the Jacobian is already in canonical form for a centre. At odd multiples of  $\pi$  it is easy to show that the Jacobian has eigenvalues  $\lambda_1 = 1$  and  $\lambda_2 = -1$ , thus these fixed points are saddles; further calculation gives the corresponding eigenvectors to be  $\underline{E_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 

and  $\underline{E}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ . Using this information we are able to hand draw the first phase diagram in Fig 5.4. To obtain the phase diagram for the nonlinear problem we are able to solve the differential equation for  $dx_2/dx_1$  as follows:

$$\frac{dx_2}{dx_1} = -\frac{\sin x_1}{x_2} \quad \Rightarrow \quad \int x_2 \, dx_2 = \int -\sin x_1 \, dx_1$$
$$\Rightarrow \quad \frac{x_2^2}{2} = \cos x_1 + C \quad \Rightarrow \quad x_2 = \pm \sqrt{2(\cos x_1 + C)} \tag{5.11}$$

This final expression generates all the trajectories in the phase plane for the nonlinear system.

Let us consider what is happening around the origin. If we consider the trajectory through the point  $P(\pi/3, 1)$  then Eq 5.11 gives C = 0 thus  $x_2 = \pm \sqrt{2} \cos x_1$ . The  $\pm$  in Eq 5.11 indicates that the trajectory is symmetric about the  $x_1$ -axis and since this particular curve cuts the  $x_1$ -axis at  $\pm \pi/2$  the complete trajectory is a closed curve. (see Fig 5.5). As we continue to consider trajectories further and further from the origin we reach a point where C = 1, for example the trajectory through the point  $Q(\pi/3, \sqrt{3})$ . This trajectory cuts the  $x_1$ -axis at  $\pm \pi$  (see Fig 5.5). Moving even further out from



Figure 5.4: Hand drawn linearised diagram; Nonlinear sine pendulm showing the centres at even multiples of  $\pi$ .



Figure 5.5:

the origin (point R) C becomes greater than 1 and therefore  $x_2$  is never zero and the trajectories are no longer closed curves (see Fig 5.5). We have thus established that the phase diagram for the nonlinear system also has a centre at the origin. Continuing the analysis we can establish that the nonlinear system has centres at all fixed points with  $x_1 = 2k\pi$ , that is to say at even multiples of  $\pi$ . The complete phase diagram is shown in Fig 5.4, the caption non damping referring to the fact that we have ignored air resistance and friction. Below we consider the sine pendulum with damping taken into account and draw the revised phase diagram.

### Example 5.5 - damped sine pendulum

This final example shows how by introducing an extra term into the equations to model the idea of friction the phase diagram changes and the centres become asymptotically stable fixed points. To model air resistance we can imagine that the faster the pendulum swings the greater the air resistance. Thus a simple assumption would be to take the resistance term to be proportional to the rate of change of theta.

Thus takeing the resistance to be equal to  $\epsilon \dot{\theta}$ , where  $\epsilon$  is a positive constant of proportionality, the

equation for the motion becomes:

$$\ddot{\theta} = -\sin\theta - \epsilon \dot{\theta}$$

which with the usual relabelling of the variables gives:

$$\dot{x}_1 = x_2 \quad \text{and} \quad \dot{x}_2 = -\sin x_1 - \epsilon x_2$$

. The fixed points are still on the  $x_1$ -axis at the same positions, that is to say at  $x_1 = n\pi$  To linearise we form the Jacobian matrix:

$$\begin{pmatrix} 0 & 1 \\ -\cos x_1 & -\epsilon \end{pmatrix} \text{ which } = \begin{pmatrix} 0 & 1 \\ -1 & -\epsilon \end{pmatrix} x_1 = 2k\pi \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & -\epsilon \end{pmatrix} x_1 = (2k-1)\pi$$

 $\underline{x_1 = (2k-1)\pi}$ 

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\epsilon - \lambda \end{vmatrix} = \lambda^2 + \epsilon \lambda - 1 = 0 \quad \Rightarrow \quad \lambda = \frac{-\epsilon \pm \sqrt{\epsilon^2 + 4}}{2}$$

For all  $\epsilon$  this give two non-zero values of  $\lambda$  each of different sign. Thus we still have a saddle point at  $x_1 = \text{odd}$  multiples of  $\pi$ .

$$x_1 = 2k\pi$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\epsilon -\lambda \end{vmatrix} = \lambda^2 + \epsilon \lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \frac{-\epsilon \pm \sqrt{\epsilon^2 - 4}}{2}$$

Depending on the value of  $\epsilon$  we obtain the following types of fixed point at  $x_1$  = even multiples of  $\pi$ .

- $\epsilon^2 4 < 0$ : this gives two complex eigenvalues with real part =  $-\epsilon/2$ , thus since  $\epsilon > 0$  the fixed point is a stable focus. The phase diagram for this case is given in Fig 5.6 where  $\epsilon$  has been taken equal to 1.
- $\epsilon^2 4 = 0$ : this gives a single eigenvalue  $= -\epsilon/2$  and hence the fixed point is an stable improper node since  $\epsilon > 0$ .
- $\epsilon^2 4 > 0$ : with  $\epsilon > 0$  this give two real negative eigenvalues, thus the fixed point is a stable node.

We note that no matter what positive value  $\epsilon$  takes the fixed points at the even multiples of  $\pi$  are stable and the pendulum will ultimately tend to rest at one of these points.

# 5.2 Linearisation theorem

In section 5.1 we saw how the linearisation of a nonlinear system about its fixed points could be used to construct the global phase diagram for the system. We also saw that not in all cases did the linearised system and the nonlinear system have the same phase diagram around a fixed point. This section sets out in the form of definitions and theorems what we can and cannot do as far as linearisation is concerned.



Figure 5.6: Damped sine pendulum:  $\ddot{\theta} = -\sin\theta - \dot{\theta}$ ; stable focii at even multiples of  $\pi$  saddles at odd multiples of  $\pi$ 

## Definition - simple fixed point

The system

$$\underline{\dot{x}} = \underline{X}(\underline{x})$$

is said to have a **simple fixed point** at  $\underline{x} = \underline{a}$  if:

- $\underline{X}(\underline{a}) = 0$ , that is to say  $\underline{x} = \underline{a}$  is a fixed point.
- The linearisation of the system about  $\underline{x} = \underline{a}$  is simple. That is to say if the linearisation is  $\underline{\dot{z}} = A\underline{z}$  then det  $A \neq 0$ .

The idea of an autonomous linear system having a simple fixed point has now been extended via the concept of linearisation to the idea of any autonomous system having a simple fixed point. We now state the main linearisation theorem:

### **Theorem - Linearisation**

In the neighbourhood of a simple fixed point of the nonlinear system

$$\underline{\dot{x}} = \underline{X}(\underline{x})$$

the phase portrait of the system and its linearisation are similar, except possibly in the case where the linearised system is a centre.

### Classification of fixed points

We use the classification of the linearised system to classify a simple fixed point of a nonlinear system, except where the linearisation is a centre.

As we have seen if the linearisation is a centre at the fixed point the nonlinear system may also be a centre in the sense that it is surrounded by closed trajectories. Alternatively the nonlinear system may also be a stable or unstable focus in the sense that the trajectories about the fixed point spiral into or out of the fixed point (next year, if you do the 2nd part of this module, you will learn more about this special cases known as *limit cycles*).

# Chapter 6

# **Non-Simple Fixed Points**

Up to this point the classification of a fixed point using the linearisation about the point has been restricted to simple fixed points. It is only for simple fixed points that the linearisation theorem is valid and even in that case only provided the linearized point is not a centre.

This chapter discusses the case of a non-simple fixed point. Even though we cannot use the linearisation theorem to strictly predict the behaviour of the associated non-linear problem the linearised phase diagrams will give some useful information.

The definition of a simple fixed point was formally given at the end of Chapter 5. If the linearisation of  $\underline{\dot{x}} = \underline{X}(\underline{x})$  about a fixed point is given by  $\underline{\dot{z}} = A\underline{z}$  then, z = 0 is a non-simple fixed point if  $\det(A) = 0$ .

# 6.1 Linear Examples

Let us start by considering a linear system of the form  $\underline{\dot{x}} = A\underline{x}$ . There are only three possible situations in which  $\det(A) = 0$ .

### **6.1.1** $\dot{x}_1 = 0, \ \dot{x}_2 = x_2$

This system of equations corresponds to

$$A = \left(\begin{array}{cc} 0 & 0\\ 0 & 1 \end{array}\right),$$

which clearly has one vanishing eigenvalue and zero determinant. The equations are easily solved by

$$x_1 = C_1, \qquad x_2 = C_2 e^t,$$

with  $C_1, C_2$  arbitrary constants. The phase diagram for this case is very simple. All trajectories correspond to constant values of  $x_1$ , that is they are straight lines, parallel to the  $x_2$  axis.

# **6.1.2** $\dot{x}_1 = x_1, \dot{x}_2 = 0$

This system of equations corresponds to

$$A = \left(\begin{array}{cc} 1 & 0\\ 0 & 0 \end{array}\right),$$
which clearly has one vanishing eigenvalue and zero determinant. The equations are easily solved by

$$x_1 = C_1 e^t, \qquad x_2 = C_2$$

with  $C_1, C_2$  arbitrary constants. The phase diagram for this case is also very simple. All trajectories correspond to constant values of  $x_2$ , that is they are straight lines, parallel to the  $x_1$  axis.

**6.1.3**  $\dot{x}_1 = 0, \dot{x}_2 = 0$ 

This system of equations corresponds to

$$A = \left(\begin{array}{cc} 0 & 0\\ 0 & 0 \end{array}\right),$$

which clearly has one vanishing eigenvalue and zero determinant. The equations are easily solved by

$$x_1 = C_1, \qquad x_2 = C_2,$$

with  $C_1, C_2$  arbitrary constants. The phase diagram for this case is the whole  $x_1 - x_2$  plane. Every point on the plane is a fixed point and there is no "flow" between fixed points. So, there are no trajectories, just fixed points. Given an initial condition, the system will for ever remain at that initial condition, since every initial condition is also a fixed point.

# 6.2 Non-Linear Example - One eigenvalue non-zero

The first example considers a problem that has both simple and non-simple fixed points. The simple fixed point is at the origin and can therefore be classified using the linearisation theorem. The phase diagram for the linear model about a non-simple fixed point is drawn and compared with the exact solution. (the problem is sufficiently simple to obtain an exact expression for all the trajectories.)

### Example 6.1

Given:

$$\dot{x}_1 = x_1(1 - 2x_1 - x_2)$$
  $\dot{x}_2 = x_2(1 - 2x_1 - x_2)$  (6.1)

The fixed points are obtained by equating each of the equations to zero:

$$x_1(1-2x_1-x_2) = 0$$
,  $x_2(1-2x_1-x_2) = 0 \Rightarrow x_1 = x_2 = 0$  or  $x_2 = 1-2x_1$ 

Thus there is an isolated fixed point at the origin and non-isolated fixed points on the line  $x_2 = 1 - 2x_1$ . To determine which of these are simple and which are not consider the Jacobian at each of the points.

The Jacobian is given by:

$$J_{(x_1,x_2)} = \begin{pmatrix} 1 - 4x_1 - x_2 & -x_1 \\ -2x_2 & 1 - 2x_1 - 2x_2 \end{pmatrix}$$

At (0,0):

$$J_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \det J_{(0,0)} = 1 \neq 0 \quad \Rightarrow \quad (0,0) \text{ simple fixed point}$$

Since (0,0) is a simple fixed points, and clearly not a centre, we can apply the linearisation theorem. Thus (0,0) is an unstable star node since the linearised matrix  $= \lambda I$ ,  $\lambda = 1 > 0$ .

At a fixed point  $(\alpha, \beta)$  on the line  $x_2 = 1 - 2x_1$ .

$$J_{(\alpha,\beta)} = \begin{pmatrix} 1 - 4\alpha - \beta & -\alpha \\ -2\beta & 1 - 2\alpha - 2\beta \end{pmatrix} = \begin{pmatrix} -2\alpha & -\alpha \\ -2\beta & -\beta \end{pmatrix} \Rightarrow \det J_{(\alpha,\beta)} = 0$$
(6.2)

Note that since  $(\alpha, \beta)$  is on the line  $x_2 = 1 - 2x_1$  the relationship  $\beta = 1 - 2\alpha$  has been used to simplify  $J_{(\alpha,\beta)}$ .

Thus any point on the line  $x_2 = 1 - 2x_1$  is a non-simple fixed point.

Consider now a solution of the problem for the exact equations for the phase paths and compare them with the results obtained using a linearisation about (0,0) and about a non-simple fixed point  $(\alpha, \beta)$ .

### 1. Exact Trajectories

This problem is sufficiently simple for us to be able to determine the exact equations for the phase diagram. From Eq(6.1) the differential equation for the trajectories gives:

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \frac{x_2(1 - 2x_2 - x_2)}{x_1(1 - 2x_2 - x_2)} = \frac{x_2}{x_1} \quad \Rightarrow \quad \int \frac{dx_2}{x_2} = \int \frac{dx_1}{x_1} \quad \Rightarrow \quad x_2 = Ax_1$$

These steps are valid for points off the axes and off the line  $x_2 = 1 - 2x_1$ . It can however be shown that both the axes are also trajectories.

Thus the phase paths consist of all straight lines through the origin, including the axes.

The phase diagram is shown in Fig. 6.1. For points above the line  $x_2 = 1 - 2x_1$  the expression  $1 - 2x_1 - x_2$  is negative. Thus for points above this line and to the right of the  $x_2$ -axis  $(x_1 > 0)$  and the first of the equations in Eq(6.1) gives  $\dot{x}_1 = x_1(1 - 2x_1 - x_2) < 0$ . Hence in this region  $x_1$  is decreasing and the trajectories are accordingly shown to approach the line of fixed points  $x_2 = 1 - 2x_1$ . A similar argument for points below the line of fixed points shows that the trajectories also tend to the line.

Finally it is worth remembering that from the linearisation theorem the origin is an unstable star node, so the idea of trajectories leaving the origin is consistent with the line of fixed points attracting trajectories.

### 2. Linearisation about (0,0) and a point $S(\alpha,\beta)$ on the line of fixed points.

About (0,0) it has already been shown that the linearisation gives an unstable star node.

Consider a point  $S(\alpha, \beta)$  on the line of fixed points, thus  $\beta = 1 - 2\alpha$  and the Jacobian is given

as above in Eq(6.2) by 
$$J_{(\alpha,\beta)} = \begin{pmatrix} -\alpha & -\alpha \\ -2\beta & -\beta \end{pmatrix}$$

The eigenvalues of  $J_{(\alpha,\beta)}$  are given by:

$$\det \begin{pmatrix} -2\alpha - \lambda & -\alpha \\ -2\beta & -\beta - \lambda \end{pmatrix} = \lambda^2 + (\beta + 2\alpha)\lambda = 0 \quad \text{since } \beta = 1 - 2\alpha \quad \Rightarrow \quad \lambda^2 + \lambda = 0 \quad \Rightarrow \quad \lambda = 0, -1$$



Figure 6.1: The fixed points are along the line  $x_2 = 1 - 2x_1$  and at the origin. By considering Eq(6.1) the direction of the trajectories can be added to the diagram. The point S is a point on the line of fixed points with coordinates  $(\alpha, \beta)$ .

The eigenvectors are calculated as:

$$\lambda_{1} = -1, \quad \begin{pmatrix} -2\alpha + 1 & -\alpha \\ -2\beta & -\beta + 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad a(1-2\alpha) - \alpha b = 0 \quad \Rightarrow \quad \frac{a}{b} = \frac{\alpha}{\beta} \quad \Rightarrow \quad \underline{E}_{1} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$
$$\lambda_{2} = 0, \quad \begin{pmatrix} -2\alpha & -\alpha \\ -2\beta & -\beta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad 2a = -b \quad \Rightarrow \quad \underline{E}_{2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Remembering that these vectors are relative to the point S the eigenline  $\langle \underline{E}_2 \rangle$  lies along the line of fixed points since they both have the same gradient, namely -2, and pass through  $(\alpha, \beta)$ . Since S has position vector  $(\alpha, \beta)$  and  $\underline{E}_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$  the eigenline  $\langle \underline{E}_1 \rangle$  will run through the

origin and S.

The linear diagram about S is then completed in the same way as in Fig. ?? above. However in this example  $\lambda_1 = -1 < 0$ , thus the trajectories, all of which are parallel to  $\underline{E}_1$ , will tend towards  $\langle \underline{E}_2 \rangle$ .

Also the trajectories about the origin will look like an unstable star node.

As seen in Fig. 6.2 it is difficult to imagine how the star node about the origin, with straight line trajectories at all angles, links up with the parallel trajectories about S.

From the exact diagram in Fig. 6.1 it can be seen that it is the linearisation about S that is at fault and indeed gets worse as points further away from S are considered.

# 6.3 Non-Linear Example - Both eigenvalues zero

The following example is taken from the Jan 2006 exam and is a good example of non-simple fixed points where the curve containing the fixed points is not a straight line.

## Example 6.2



Figure 6.2: The diagram shows the phase paths drawn according to the linearisation about the simple fixed point at the origin and the linearisation about the non-simple fixed point S. In this example the non-simple fixed points of the non-linear system coincide with the non-simple fixed points of the linearised system about S. In general this will not be the case.

Consider the system:

$$\dot{x}_1 = x_2(1 - x_1^2 - x_2^2) = 0$$
  $\dot{x}_2 = x_1(x_1^2 + x_2^2 - 1) = 0$  (6.3)

The fixed points are obtained by solving:

$$x_2(1 - x_1^2 - x_2^2) = 0$$
 (i)  $x_1(x_1^2 + x_2^2 - 1) = 0$  (ii)

If  $(1 - x_1^2 - x_2^2) = 0$  both equations are satisfied, thus all points on the unit circle are fixed points.

Also (i) 
$$\Rightarrow$$
  $x_2 = 0$ , thus (ii)  $\Rightarrow$   $x_1(x_1^2 - 1) = 0 \Rightarrow$   $x_1 = 0, x_1 = \pm 1.$ 

Since  $(\pm 1, 0)$  are on the unit circle we deduce that the only fixed points are on the unit circle or at the origin.

Construct the Jacobian:

$$J = \begin{pmatrix} -2x_1x_2 & 1 - x_1^2 - 3x_2^2 \\ 3x_1^2 + x_2^2 - 1 & 2x_1x_2 \end{pmatrix}$$

**At** (0,0):

$$\det J = \det \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} = 1 \neq 0$$

Thus the origin is a simple fixed point.

$$J_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{canonical form for centre}$$

Thus although the origin is simple it is not possible to apply the linearisation theorem since the linear problem is a centre.

#### On the unit circle:

Let  $(\alpha, \beta)$  be the coordinates of a point S on the unit circle: thus  $\alpha^2 + \beta^2 = 1$ . The Jacobian is now given by:

$$J_{(\alpha,\beta)} = \begin{pmatrix} -2\alpha\beta & 1-\alpha^2 - 3\beta^2 \\ 3\alpha^2 + \beta^2 - 1 & 2\alpha\beta \end{pmatrix} = \begin{pmatrix} -2\alpha\beta & -2\beta^2 \\ 2\alpha^2 & 2\alpha\beta \end{pmatrix} \quad \Rightarrow \quad \det J_{(\alpha,\beta)} = 0 \tag{6.4}$$

Thus the unit circle consists of non-simple fixed points.

1. Exact trajectories Since it is not possible to apply the linearisation theorem to any of the fixed points the linear description may well be a poor representation of the true phase diagram. Again the problem is sufficiently simple to be able to obtain exactly the equations of the trajectories.

From Eq(6.3):

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \frac{x_1(x_1^2 + x_2^2 - 1)}{x_2(1 - x_1^2 - x_2^2)} = -\frac{x_1}{x_2} \quad \Rightarrow \quad \int x_2 \, dx_2 = -\int x_1 \, dx_1 \quad \Rightarrow \quad x_2^2 + x_1^2 = C$$

Thus all the trajectories are circles centre the origin with the origin and the unit circle being fixed points. To determine the orientation of the trajectories consider the sign of  $\dot{x}_1$  in the region above the  $x_1$  axis, that is to say where  $x_2 > 0$ .

From Eq(6.3),  $\dot{x}_1 = x_2(1 - x_1^2 - x_2^2)$ . Outside the unit circle  $(1 - x_1^2 - x_2^2) < 0$  thus  $\dot{x}_1 < 0$  and hence  $x_1$  is decreasing. This implies the trajectories have an anti-clockwise orientation. Similarly inside the unit circle the trajectories have a clockwise orientation.

It is worth noting that in this example both the exact and linear solutions give a centre at the origin. As we have seen in Chapter 5, Example 5.3 this is not always the case.

The exact phase diagram is given in Fig 6.3



Figure 6.3: Fixed points at the origin and unit circle centre the origin.

Fig. ??.

## 2. Linearisation about a point $S(\alpha, \beta)$ on the unit circle of fixed points.

Using  $J_{(\alpha,\beta)}$  from Eq(6.4) above the eigenvalues are given by:

$$\det \begin{pmatrix} -2\alpha\beta - \lambda & -2\beta^2 \\ 2\alpha^2 & 2\alpha\beta - \lambda \end{pmatrix} = \lambda^2 = 0$$

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Thus the two eigenvalues are both zero. The single eigenvector  $\underline{E}_1$  is given by:

$$\begin{pmatrix} -2\alpha\beta & -2\beta^2 \\ 2\alpha^2 & 2\alpha\beta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad \frac{a}{b} = \frac{-\beta}{\alpha} \quad \Rightarrow \quad \underline{E}_1 = \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$$

The first thing to note is that since S has position vector  $(\alpha, \beta)$  the line from the origin to S is perpendicular to  $\underline{E}_1$ .

Denoting as usual the linearised coordinates by  $\underline{z}$ , the linearisation about S is given by:

$$\underline{\dot{z}} = \begin{pmatrix} -2\alpha\beta & -2\beta^2 \\ 2\alpha^2 & 2\alpha\beta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Rightarrow \begin{aligned} \dot{z}_1 &= -2\alpha\beta z_1 - 2\beta^2 z_2 \\ \dot{z}_2 &= 2\alpha^2 z_1 + 2\alpha\beta z_2 \end{aligned} \Rightarrow \\ \dot{z}_1 &= -2\beta(\alpha z_1 + \beta z_2) \\ (6.5)$$

Since  $\underline{E}_1 = \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$  is relative to the  $\underline{z}$  coordinates through S it lies along the line  $\alpha z_1 + \beta z_2 = 0$ . Thus for points above the line of fixed points defined by  $\underline{E}_1$ ,  $\alpha z_1 + \beta z_2 > 0$  and hence from Eq(6.5)  $\dot{z}_1 < 0$ . Hence  $z_1$  is decreasing and the trajectories above the line of fixed points points from right to left. Similarly the trajectories below the line of fixed point run in the opposite direction. See