MA2605

CITY UNIVERSITY

London

BSc Honours Degree in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
Mathematics and Finance

Part 2

Dynamical Systems

2011

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.
All necessary working must be shown.

1. Consider the following first order differential equation:

$$\frac{dy}{dt} = \dot{y} = \cos(y). \tag{1}$$

(a) Is the equation above linear? Is the equation above autonomous? In both cases, justify your answers. **[2]**

(b) Find and classify all the fixed points of Eq. (1). **[6]**

(c) Draw the phase diagram associated to Eq. (1) in the region $-3\pi < y < 3\pi$. **[3]**

(d) The linearisation technique allows us to approximate a generic equation of the form

$$\frac{dy}{dt} = X(y), \tag{2}$$

by

$$\frac{dz}{dt} = X'(a)z \quad \text{with} \quad z = y - a, \tag{3}$$

near a fixed point $y = a$. Explain how one may obtain Eq. (3) starting with Eq. (2). **[3]**

(e) Write down the solution to Eq. (3) that satisfies the initial condition $y = y_0$ for $t = t_0$. **[3]**

(f) Use the result of sections (d) and (e) to find a linear approximation to the solution of Eq. (1) near the fixed point $y = -\frac{\pi}{2}$ which satisfies the initial condition $y = 1$ for $t = 1$. **[3]**

(g) Solve Eq. (1) with initial conditions $y = 0$ for $t = 1$. Indicate the interval I of values of t for which this solution is defined.
Hint: you may want to use the change of variables $x = \tan\frac{y}{2}$ and the identities $\cos^2\frac{y}{2} = \frac{1}{1+\tan^2\frac{y}{2}}$ and $\cos y = \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}$. **[5]**

2. Consider the following system of linear differential equations:

$$\frac{dx_1}{dt} = \dot{x}_1 = 2x_1 - 5x_2 + 1, \qquad \frac{dx_2}{dt} = \dot{x}_2 = x_1 + 4x_2. \tag{4}$$

(a) Write the equations in the standard form $\dot{\underline{x}} = A\underline{x} + \underline{b}$. **[1]**

(b) The system has a single fixed point. Find the fixed point. Is it a simple fixed point? Justify your answer. **[3]**

(c) Compute the eigenvalues and eigenvectors of the matrix A. Hence classify the fixed point. **[5]**

(d) Define the vector $\underline{z}$ in terms of which the equation can be re-written in the form $\dot{\underline{z}} = A\underline{z}$. **[1]**

(e) Write down the Jordan normal form J of the matrix A and the matrix P which enters the relation $J = P^{-1}AP$. Write down the relationship between the vector $\underline{z}$ above and the vector $\underline{y}$ which solves the canonical equation $\dot{\underline{y}} = J\underline{y}$. **[5]**

(f) Write down the general solution for the vectors $\underline{x}$ and $\underline{y}$. **[5]**

(g) Sketch the phase diagram of the system in the $z_1 - z_2$ plane clearly indicating its main features. **[5]**

3. Consider the following system of first order nonlinear differential equations:

$$\frac{dx_1}{dt} = \dot{x}_1 = 2 - x_1 - x_2, \qquad \frac{dx_2}{dt} = \dot{x}_2 = 2x_2^2 e^{x_1}(2 - x_1 - x_2). \tag{5}$$

(a) Show that the fixed points of the system fall on a line. **[3]**

(b) Calculate the Jacobian matrix associated to Eq. (5) for a general point (x_1, x_2). **[5]**

(c) Use the result of (b) to determine whether or not the fixed points of section (a) are simple. **[5]**

(d) By dividing the two equations in (5) obtain a new equation of the form $\frac{dx_2}{dx_1} = X(x_1, x_2)$. Find the general solution to this equation. Find the particular solution which satisfies the initial conditions $x_2 = -\frac{1}{4}$ if $x_1 = 0$. **[5]**

(e) Sketch the particular solution found in section (d) (the one with initial conditions $x_2 = -\frac{1}{4}$ if $x_1 = 0$) in the $x_1 - x_2$ plane. Add arrows to your plot, as you would do for a phase space diagram. Does the direction of the arrows change when the trajectory above meets the line of fixed points obtained in (a)? Why? **[7]**

4. Consider the following second order linear differential equation:

$$\ddot{x} - 6\dot{x} + 8x = 0, \tag{6}$$

where, as usual, $\ddot{x} = \frac{d^2x}{dt^2}$ and $\dot{x} = \frac{dx}{dt}$.

(a) Find a suitable change of variables that transforms Eq. (6) into a pair of first order linear differential equations for variables x_1, x_2. **[3]**

(b) Using the result of (a) write equation (6) in the standard matrix form $\dot{\underline{x}} = A\underline{x}$. **[2]**

(c) Find the eigenvalues and eigenvectors of the matrix A. Thus write down the matrix J that gives the Jordan normal form of A. **[5]**

(d) Classify the nature of the fixed point at the origin. **[3]**

(e) Find the general solution of the canonical system of equations $\dot{\underline{y}} = J\underline{y}$. Find a particular solution for $\underline{x}$ which satisfies the initial condition $\underline{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ at $t = 0$. **[6]**

(f) Sketch the phase diagrams in the $y_1 - y_2$ plane and in the $x_1 - x_2$ plane clearly indicating their main features. **[6]**

5. Consider the following first order nonlinear differential equations

$$\frac{dx_1}{dt} = \dot{x}_1 = x_1^2 - \frac{\pi^2}{4}, \qquad \frac{dx_2}{dt} = \dot{x}_2 = \sin x_1 - \cos x_2. \tag{7}$$

(a) Find the fixed points of the system. **[6]**

(b) Find the general form of the Jacobian matrix of the system and evaluate it only at the fixed points with $0 < x_2 < 3\pi$. **[6]**

(c) Classify the fixed points in (b) according to their linearisation. **[4]**

(d) Solve the linearised equations for the same fixed points considered in (b) and (c). **[6]**

(e) Explain whether or not you think that the phase space diagram near the fixed points will look similar to the the phase space diagram of the linearised version of the system near the fixed points. **[3]**

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