

Sheet 3: Second order linear, autonomous systems

1. Find the eigenvalues and eigenvectors of the following 2×2 matrices:

$$M_1 = \begin{pmatrix} 13 & 9 \\ -18 & -14 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 5 & 4 \\ -10 & 7 \end{pmatrix}, \quad \text{and} \quad M_3 = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

Hence, in each case, find a non-singular matrix P such that $J = P^{-1}AP$ with A being one of the matrices above, and J a Jordan real normal form of A .

2. You may have noticed that the matrices above are precisely the same matrices involved in the systems of equations that we looked at in question 4, sheet 1. Write each of those systems of equations in matrix form and, in each case,

- (a) classify the fixed point at the origin,
- (b) sketch the phase diagram,
- (c) find the general solution as a function of the variable t

3. For each of the following systems of equations:

$$\begin{aligned} (a) \quad & \dot{x}_1 = -17x_1 + 39x_2 + 13, \quad \dot{x}_2 = -6x_1 + 13x_2 + 26, \\ (b) \quad & \dot{x}_1 = 6x_2 + 6, \quad \dot{x}_2 = -x_1 + 5x_2 + 1 \\ (c) \quad & \dot{x}_1 = -4x_1 + 9x_2 + 2, \quad \dot{x}_2 = -x_1 + 2x_2 + 1 \end{aligned}$$

- Write the system in the form $\dot{\underline{x}} = A\underline{x} + \underline{b}$.
- Find the fixed point.
- Change variables so that you can rewrite the system in the form $\dot{\underline{z}} = A\underline{z}$ where $\underline{z} = \underline{x} - \underline{a}$ and \underline{a} is the fixed point obtained before.
- Find the eigenvalues and eigenvectors of A .
- Classify the fixed point.
- Sketch the phase diagram.
- Write down the general solution for $\underline{x}(t)$.

4. Consider the equation $\dot{\underline{x}} = A\underline{x}$ with

$$A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \text{and} \quad x \in [0, \pi].$$

- Show that the fixed point at the origin is simple.
- Calculate the eigenvalues of A . Hence determine the values of θ for which the origin is:
 - (a) a stable star node
 - (b) an unstable star node
 - (c) a centre
 - (d) a stable focus
 - (e) an unstable focus