**Sheet 4: nonlinear two dimensional systems**

1. Consider the following three systems of of nonlinear first-order differential equations:

$$\dot{x}_1 = X_1(x_1, x_2), \qquad \dot{x}_2 = X_2(x_1, x_2),$$

with

| | | |
|---|---|---|
| (a) | $X_1(x_1, x_2) = x_1^2 x_2 + x_1 x_2$ | $X_2(x_1, x_2) = \sin x_1 \cos x_2$ |
| (b) | $X_1(x_1, x_2) = x_1^2 x_2^2 + x_1 \sin x_2 + 1 - e^{x_1}$ | $X_2(x_1, x_2) = x_1 \sin x_2 + x_2 \cos x_1$ |
| (c) | $X_1(x_1, x_2) = x_1 + x_2 + x_1 x_2$ | $X_2(x_1, x_2) = x_1 \sin x_2 + x_2 \cos x_1$ |

(a) Write down the generic Jacobian matrix for each of the three systems.

(b) Show that all three systems have a fixed point at the origin.

(c) Write down the linearisation of each of the three systems about the fixed point at the origin.

(d) For each system investigate whether or not the fixed point at the origin is simple.

(e) In the case(s) when this fixed point is simple, classify its nature from the linearised equations.

2. Given the equations

$$\dot{x}_1 = x_1 \cos x_2, \qquad \dot{x}_2 = x_2 \cos x_1,$$

with $-\pi < x_1 < \pi$ and $-\pi < x_2 < \pi$,

(a) Locate the fixed points in the given region.

(b) Write down the Jacobian for the system.

(c) Write down the linearisation of the system about each of its fixed points.

(d) Classify each of the fixed points.

(e) By considering the equation for $\frac{dx_2}{dx_1}$ show that $x_2 = \pm x_1$ are both solutions to this equation and therefore they are trajectories in the phase space diagram of the system.

(f) Show that the positive $x_1$-axis is also a trajectory for $x_1 = C_1 e^t$ and $C_1 > 0$ and the negative $x_1$-axis is a trajectory with $x_1 = C_1 e^t$ and $C_1 < 0$.

(g) Carefully sketch the phase diagram, using the eigenvectors of the linearised problem whenever possible.

3. Consider the equation

$$\ddot{x} - \dot{x}(1 - 3x^2 - 2\dot{x}^2) + x = 0.$$

(a) By defining $x_1 = x$ and $x_2 = \dot{x}_1$, rewrite the equation as a pair of 1st order differential equations, one of which is linear and one nonlinear.

(b) Linearise the system of equations about the single fixed point at the origin.

(c) Classify the fixed point, solve the linearised equations and draw the corresponding phase diagram.

(d) It turns out that the phase diagram that you obtain from linearisation is not quite what is really happening in the nonlinear system. Use the software that is available in the course web page to produce a vector field diagram associated to the original equations.

(e) From this diagram deduce the behaviour of phase space trajectories as $t \to \infty$. You should find that this behaviour differs from the one you would predict from linearisation.