Solutions to Sheet 1: differential equations and phase diagrams

1. For the first equation we have:

$$\frac{dy}{dx} = 1 + y^2 \quad \Rightarrow \quad \int \frac{dy}{1+y^2} = \int dx \quad \Rightarrow \quad \tan^{-1}(y) = x + C,$$

with C the integration constant. If we want that $y(0) = 1$ we have that:

$$\tan^{-1}(1) = C \quad \Rightarrow \quad \frac{\pi}{4} = C.$$

and the solution is

$$y(x) = \tan\left(x + \frac{\pi}{4}\right).$$

Similarly for the other equations,

$$\frac{dy}{dx} = x\cos^2 y \quad \Rightarrow \quad \int \frac{dy}{\cos^2 y} = \int x dx \quad \Rightarrow \quad \tan y = \frac{x^2}{2} + C.$$

The initial condition $y(0) = \frac{\pi}{4}$ gives $C = 1$ so that

$$y(x) = \tan^{-1}\left(\frac{x^2}{2} + 1\right).$$

For the last equation let us now include the initial conditions as integration limits instead,

$$\frac{dy}{dx} = y^2+2y-3 \quad \Rightarrow \quad \int_{-1}^{y} \frac{dy}{y^2+2y-3} = \int_0^x dx \quad \Rightarrow \quad \int \frac{dy}{(y-1)(y+3)} = x+C \quad \Rightarrow$$

$$\frac{1}{4}\int_{-1}^{y}\left(\frac{1}{y-1} - \frac{1}{y+3}\right) dy = x \quad \Rightarrow \quad \ln\left(\frac{y-1}{y+3}\right)\Bigg|_{-1}^{y} = \ln\left(\frac{(y-1)2}{(y+3)(-2)}\right) = \ln\left(\frac{1-y}{y+3}\right) = 4x.$$

$$\frac{y-1}{y+3} = e^{4x} \quad \Rightarrow \quad y = \frac{1-3e^{4x}}{1+e^{4x}}.$$

2. The method of integrating factors tells us that if we have an equation such as:

$$\frac{dy}{dx} + P(x)y = Q(x),$$

then it can be solved by defining the integrating factor $R(x) = e^{\int P(x)dx}$. The solution is then,

$$y(x) = \frac{1}{R(x)}\left(\int R(x)Q(x)dx + C\right).$$

The first equation was

$$\frac{dy}{dx} = \cos x - y\tan x, \qquad \text{with} \qquad y(0) = 1.$$

Here

$$R(x) = e^{\int \tan x dx} = e^{-\ln(\cos x)} = \frac{1}{\cos x}.$$

Therefore

$$y(x) = \cos x \left(\int dx + C \right) = (x + C) \cos x.$$

The initial condition fixes $C = 1$.

For the second equation we have,

$$\frac{dy}{dx} = e^x - 3y, \qquad \text{with} \qquad y(0) = \frac{1}{2}.$$

We compute,

$$R(x) = e^{\int 3dx} = e^{3x}.$$

Therefore

$$y(x) = e^{-3x} \int e^{3x} e^x dx + Ce^{-3x} = \frac{e^x}{4} + Ce^{-3x}.$$

The initial condition fixes $C = \frac{1}{4}$.

Finally, for the last equation

$$\frac{dy}{dx} = \cos x - y \cot x, \qquad \text{with} \qquad y(\frac{\pi}{2}) = 1.$$

The function $R(x)$ is

$$R(x) = e^{\int \cot x dx} = e^{\ln \sin x} = \sin x.$$

Thus,

$$y(x) = \frac{1}{\sin x} \int \sin x \cos x dx + \frac{C}{\sin x} = \frac{\sin x}{2} + \frac{C}{\sin x}.$$

The initial condition fixes $C = 1/2$.

3. The phase diagram is

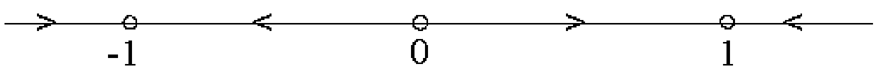


Figure 1: The phase space diagram for $\dot{y} = y(1 - y^2)$.

We can clearly see that the equation has three fixed points corresponding to

$$\frac{dy}{dt} = y(1 - y^2) = 0, \quad \Rightarrow \quad y = 0, 1, -1.$$

In order to find out whether $y(t)$ is increasing or decreasing in each region of phase space we have to look at the sign of the derivative. We have four regions:

- The region $y > 1$, has $\dot{y} < 0$, that is $y(t)$ decreases for increasing time.
- The region $0 < y < 1$ has $\dot{y} > 0$, so $y(t)$ increases as time increases.
- The region $-1 < y < 0$ has $\dot{y} < 0$, so $y(t)$ decreases as time increases.
- Finally, if $y < -1$, $\dot{y} > 0$, so $y(t)$ increases as time increases.

The phase diagram tells us a lot about how the solution of the differential equation should behave. The phase diagram tells us that our solution should behave in four different ways, depending on the initial condition:

- If the initial condition, y_0 is $y_0 > 1$ we know that $y(t)$ decreases with time. So our solution $y(t)$ will start at the value y_0 and then decrease until reaching the value 1 where it will stay (asymptotic value).
- If the initial condition is $0 < y_0 < 1$, the phase diagram tells us that it will evolve towards the value 1 as well.
- If the initial condition is $-1 < y_0 < 0$, the phase diagram tells us that it will evolve towards the value -1, decreasing with time.
- Finally, if $y_0 < -1$, it will also evolve towards the value -1.

In summary all solutions with initial condition $y_0 > 0$ evolve toward the fixed point value 1 and all solutions with initial condition $y_0 < 0$ evolve towards the fixed point value -1.

Although the problem did not ask for this, it is interesting to try to solve the equation explicitly. Let the initial condition be $y_0 = y(t_0)$. The general solution is

$$\int_{y_0}^{y} \frac{dy}{y(1-y^2)} = \int_{t_0}^{t} dt \quad \Rightarrow \quad \int \left(\frac{1}{y} + \frac{1}{2(1-y)} - \frac{1}{2(1+y)}\right) dy = t - t_0.$$

Therefore

$$\ln y - \frac{1}{2}\ln(1-y^2) - \ln y_0 + \frac{1}{2}\ln(1-y_0^2) = t - t_0 \quad \Rightarrow \quad \frac{y}{y_0}\sqrt{\frac{1-y_0^2}{1-y^2}} = e^{t-t_0}.$$

Squaring both sides and solving for y we obtain

$$\frac{y^2(1-y_0^2)}{y_0^2(1-y^2)} = e^{2(t-t_0)} \quad \Rightarrow \quad y^2\left(1 - y_0^2 + y_0^2 e^{2(t-t_0)}\right) = y_0^2 e^{2(t-t_0)} \quad \Rightarrow \quad y^2 = \frac{y_0^2 e^{2(t-t_0)}}{1 - y_0^2 + y_0^2 e^{2(t-t_0)}}.$$

As we can see there are really two solutions, depending on whether we choose the positive or the negative sign when taking the square root. Our choice will depend on the initial conditions.

Let us consider each region in the phase diagram separately:

(a) Region $y_0 < -1$: In this case y_0 is negative and therefore the solution that we need to pick is

$$y = -\sqrt{\frac{y_0^2 e^{2(t-t_0)}}{1 - y_0^2 + y_0^2 e^{2(t-t_0)}}}.$$

This function is defined everywhere, except if the denominator vanishes. This can happen for some value of $t = t_\infty$ if $y_0^2 - 1 = y_0^2 e^{2(t_\infty - t_0)}$, which gives

$$t_\infty = t_0 + \frac{1}{2}\log\left(\frac{y_0^2 - 1}{y_0^2}\right).$$

The number inside the argument of the log function is $1 - 1/y_0^2$, which is smaller than 1. So the contribution from the log is negative, which means that $t_\infty < t_0$. Therefore the interval of definition of the solution would be $I \in (t_\infty, \infty)$. Concerning the behaviour of the function, we can see that $y \to -1$ when $t \to \infty$, which is what one would expect from the phase diagram.

(b) Region $-1 < y_0 < 0$: In this case y_0 is still negative and therefore the solution that we need to pick is the same as above

$$y = -\sqrt{\frac{y_0^2 e^{2(t-t_0)}}{1 - y_0^2 + y_0^2 e^{2(t-t_0)}}}.$$

This function is defined everywhere, except if the denominator vanishes. This can not happen in this region, as the denominator remains always positive. Therefore $I \in (-\infty, \infty)$.

$y \to -1$ when $t \to \infty$, which is what one would expect from the phase diagram. Also, $y \to 0$ when $t \to -\infty$, which is also consistent.

(c) Region $0 < y_0 < 1$: In this case $y_0 > 0$ and therefore the solution that we need to pick is

$$y = \sqrt{\frac{y_0^2 e^{2(t-t_0)}}{1 - y_0^2 + y_0^2 e^{2(t-t_0)}}}.$$

This function is defined everywhere, except if the denominator vanishes. This can not happen in this region, as the denominator remains always positive. Therefore $I \in (-\infty, \infty)$.

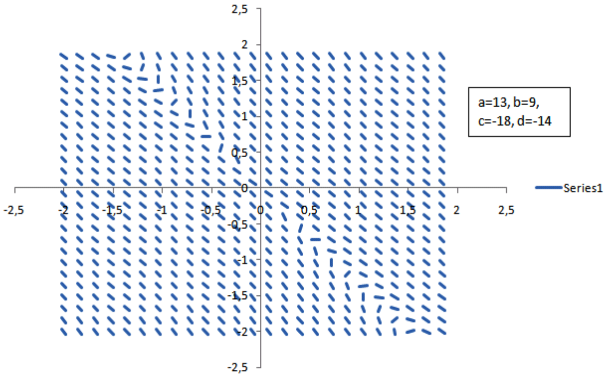
$y \to 1$ when $t \to \infty$, which is what one would expect from the phase diagram. Also, $y \to 0$ when $t \to -\infty$, which is also consistent.

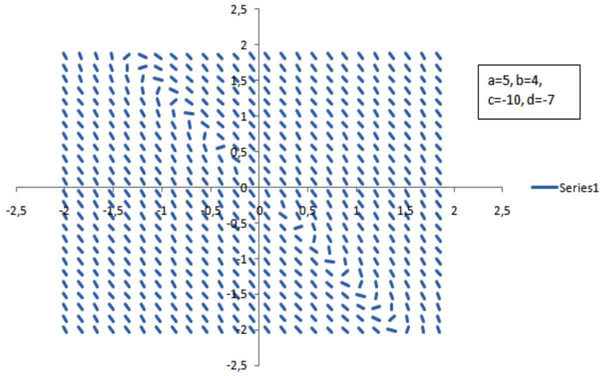
(d) Region $y_0 > 1$: In this case $y_0 > 0$ and therefore the solution that we need to pick is

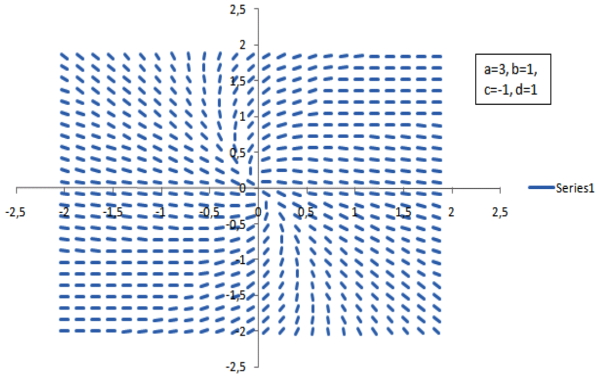
$$y = \sqrt{\frac{y_0^2 e^{2(t-t_0)}}{1 - y_0^2 + y_0^2 e^{2(t-t_0)}}}.$$

This function is defined everywhere, except if the denominator vanishes. This happens for the same t_∞ computed for the first case and the log term is again negative in this case, which means that $I \in (t_\infty, \infty)$. $y \to 1$ when $t \to \infty$ as expected.

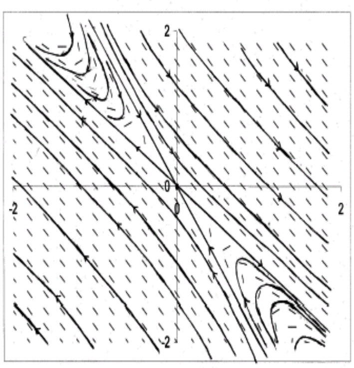
4. The velocity/vector diagrams for the three sets of equations are:

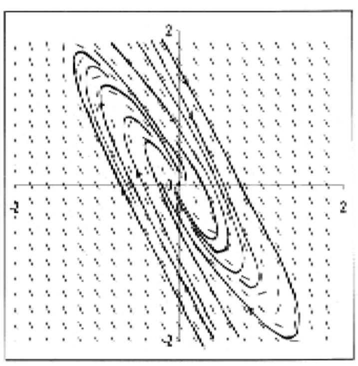


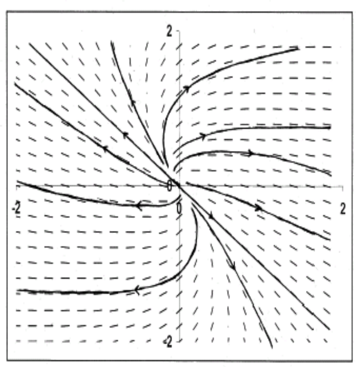




If we try to draw the phase diagrams for the three cases on top of the vector diagram we obtain the three plots below. Some features of these phase diagrams can indeed be seen from the vector diagrams above if one pays close attention to the change in slope of neighboring segments. However, it is hard to deduce all features just from the vector diagram, as we have no way to know which segments lie on the same phase space trajectory. Also, we still need to use the original equations to find out what the direction of the arrows will be.







Fortunately there is a general method that allows one to know precisely what the phase space diagrams of equations of the type considered here will look like. For equations which are linear both in x and y and have a fixed point at $(0,0)$ one can precisely classify all types of phase diagrams that are possible, depending on the values of a, b, c and d. The core part of this module will precisely deal with understanding this method.

According to this classification, it turns out that for the first picture above, the origin is a fixed point of a type known as a saddle point. For the second case the origin is what is called an stable focus. In this case all phase space trajectories spiral in towards the origin when $t \to \infty$. Finally, in the third diagram the origin is a fixed point of a type know as an improper unstable node. In this case unstable means that all trajectories tend to flow away from the origin as t increases (the opposite behaviour as for the focus before).