Solutions to Sheet 2: phase diagrams of 1-dimensional systems and linearisation

1. The fixed points for the various equations are, by definition, the points at which $\frac{dy}{dx} = 0$. They are as follows:

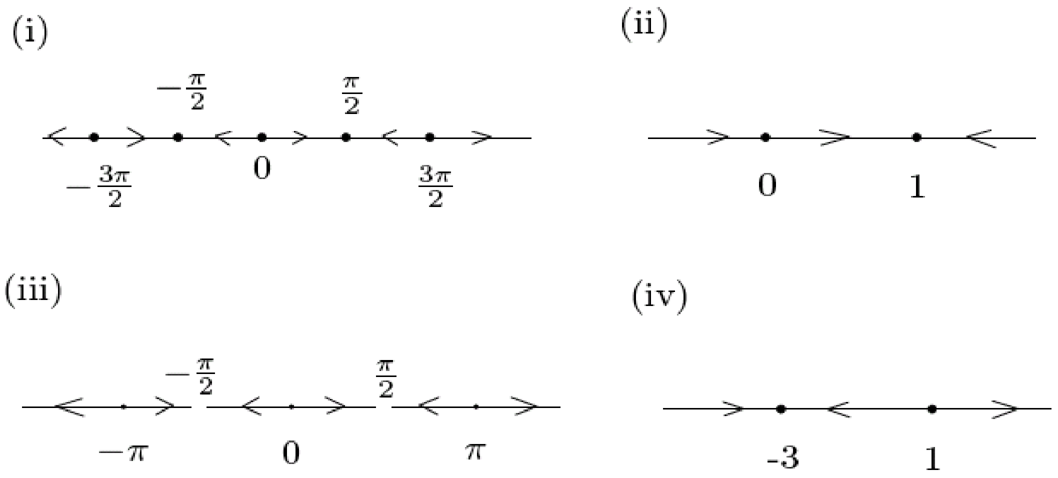
   (i) The solutions to $y\cos(y) = 0$ are: $y = 0$ and $y = \frac{(2k+1)\pi}{2}$ for $k = 0, \pm1, \pm2...$

   (ii) The solutions to $y^2(1-y) = 0$ are: $y = 0$ and $y = 1$.

   (iii) The solutions to $\tan(y) = 0$ are: $y = k\pi$ for $k = 0, \pm1, \pm2...$

   (iv) The solutions to $y^2 + 2y - 3 = 0$ are: $y = 1$ and $y = -3$.

   By investigating the sign of $\frac{dy}{dx}$ in the regions between the various fixed points you should find the following phase space diagrams:



   Therefore the various fixed points can be classified as follows:

   (i) The fixed point at $y = 0$ is a repellor. All fixed points of the form $\pm\frac{(4k+1)\pi}{2}$ with $k = 0, 1, 2...$ are also repellors. All other fixed points, which are of the form $\pm\frac{(4k+3)\pi}{2}$ with $k = 0, 1, 2...$ are attractors.

   (ii) The point $y = 0$ is a shunt. The point $y = 1$ is a attractor.

   (iii) In this case the phase diagram has some gaps at the points $y = \frac{(2k+1)\pi}{2}$ for $k = 0, \pm1, \pm2...$ since the function $\tan(y)$ is not defined at those points. All the fixed points are repellors.

   (iv) The point $y = -3$ is an attractor. The point $y = 1$ is a repellor.

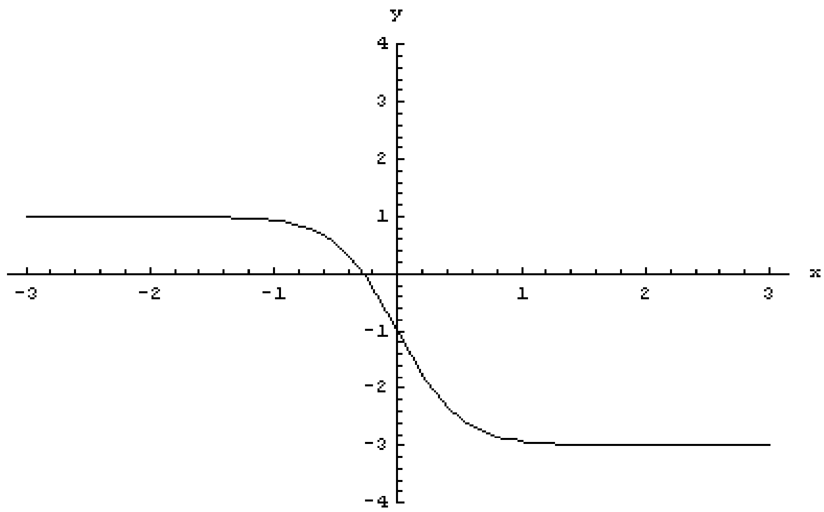
   You can also carry out this classification by looking at the sign of the derivative of r.h.s. of the equations at the fixed point (the criterium that we use when we linearise about the fixed point).

2. The solution to this equation with the given initial condition was computed in the previous exercise sheet. It was:

$$y(x) = \frac{1-3e^{4x}}{1+e^{4x}}.$$

This function is well defined for all values of x since the denominator never vanishes. Therefore $I = (-\infty, \infty)$.

Before presenting the plot of the function above, it is in fact quite easy to figure out how this function look. For example, when $x \to \infty$, $y \to -3$, whereas for $x \to -\infty$, $y \to 1$. In addition $y(0) = -1$ and the function becomes zero when the numerator vanishes, that is at $e^{4x} = 1/3$ which gives $x = -\frac{1}{4}\ln(3) \approx -0.274653$. The linearisation of this equation about a generic fixed point $y = a$ is obtained by



the standard procedure. Consider the equation $\frac{dy}{dx} = X(y)$ and expand the r.h.s. of it in a Taylor expansion about the point $y = a$. We will do the expansion only up to first order in powers of y:

$$\frac{dy}{dx} = X(a) + X'(a)(y-a) + \cdots$$

if a is a fixed point, then $X(a) = 0$ so, at first order the equation becomes:

$$\frac{dy}{dx} = X'(a)(y-a),$$

and if we introduce the new variable $z = y - a$, then the equation becomes

$$\frac{dz}{dx} = X'(a)z.$$

For the equation at hand we need to compute $X'(y) = 2y + 2$. Therefore, for a fixed point $y = a$ the linearized equation becomes simply:

$$\frac{dz}{dx} = 2(a+1)z,$$

which is solved by

$$z = Ae^{2(a+1)x} \quad \text{so} \quad y = a + Ae^{2(a+1)x}.$$

for A a constant which is fixed by the initial conditions. At $a = -3$ with $y(0) = -1$ we have the linearised solution:

$$y = -3 + 2e^{-4x},$$

whereas for $y = 1$, with the same initial condition we have

$$y = 1 - 2e^{4x}.$$

If we look at the sign of $X'(1) = 4 > 0$, this tells us that $y = 1$ is a repellor. Similarly $X'(-3) = -4 < 0$, and therefore $y = -3$ is an attractor.

If we now plot these two linearised solutions together with the exact solution from before we get