Solutions to Sheet 3: Second order linear, autonomous systems

1. In order to find the eigevalues we need to find the zeroes of the characteristic polynomial of each of the matrices. For a matrix M this is obtained by computing the determinant

$$|M - \lambda I| = 0,$$

where I is the identity matrix. The eigevectors are the solutions to

$$M\underline{E} = \lambda \underline{E},$$

where λ is an eigenvalue of M. We now apply these definitions to each of the given matrices.

(a) For

$$M_1 = \begin{pmatrix} 13 & 9 \\ -18 & -14 \end{pmatrix},$$

the characteristic polynomial is:

$$(13 - \lambda)(-14 - \lambda) + 162 = -182 + \lambda^2 + \lambda + 162 = \lambda^2 + \lambda - 20 = 0,$$

which gives eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -5$
The first eigenvector is the solution to

$$\begin{pmatrix} 13 & 9 \\ -18 & -14 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix},$$

that is $13a + 9b = 4a$ and $-18a - 14b = 4b$. Again, the two equations are not independent, which means that we can just fix one of the constants and then obtained the other from any of the equations. For example, taking $a = 1$ we get $b = -1$. Thus the eigenvector is

$$\underline{E}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In this case, the eigenvalues are real and different from each other and the Jordan form is simply

$$J = \begin{pmatrix} -5 & 0 \\ 0 & 4 \end{pmatrix}.$$

Similarly, the eigenvector associated to eigenvalue $\lambda_2 = -5$ is the solution to

$$\begin{pmatrix} 13 & 9 \\ -18 & -14 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -5 \begin{pmatrix} a \\ b \end{pmatrix},$$

that is $13a + 9b = -5a$ and $-18a - 14b = -5b$. As usual, the two equations are not independent, which means that we can just fix one of the constants and then obtained the other from any of the equations. For example, taking $a = 1$ we get $b = -2$. Thus the first eigenvector is

$$\underline{E}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The matrix P can be constructed in terms of the eigenvectors above as

$$P = (\underline{E}_1, \underline{E}_2) = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}.$$

(b) Similarly for M_2,

$$M_2 = \begin{pmatrix} 5 & 4 \\ -10 & 7 \end{pmatrix},$$

the characteristic polynomial is

$$(5 - \lambda)(7 - \lambda) + 40 = 35 + \lambda^2 - 12\lambda + 40 = \lambda^2 - 12\lambda + 75 = 0,$$

and the eigenvalues are $\lambda_1 = 6 + i\sqrt{39}$ and $\lambda_2 = 6 - i\sqrt{39}$. Since the eigenvalues are complex conjugated to each other, it follows that the eigenvectors will also be. This can be proven in general. If E_1 is the eigenvector associated to eigenvalue λ_1 we have that

$$M_2\underline{E}_1 = \lambda_1 \underline{E}_1.$$

Complex conjugating the equation we get

$$M_2(\underline{E}_1)^* = (\lambda_1)^*(\underline{E}_1)^*,$$

which means that $\lambda_2 = (\lambda_1)^*$ is an eigenvalue with eigenvector $\underline{E}_2 = (\underline{E}_1)^*$.
Therefore, we only have to find $\underline{E}_1$,

$$\begin{pmatrix} 5 & 4 \\ -10 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (6 + i\sqrt{39})\begin{pmatrix} a \\ b \end{pmatrix}.$$

This gives equations $5a + 4b = (6 + i\sqrt{39})a$ and $-10a + 7b = (6 + i\sqrt{39})b$, which can be rewritten as $-a+4b = i\sqrt{39}a$ and $-10a+b = i\sqrt{39}b$. These equations seem different, but in fact are proportional to each other. Therefore they are not independent and we can just solve them by fixing one of the constants and then solving for the other one. Taking $a = 1$ we obtain $b = \frac{1+i\sqrt{39}}{4}$. Hence the eigenvectors are

$$\underline{E}_1 = \begin{pmatrix} 1 \\ \frac{1+i\sqrt{39}}{4} \end{pmatrix}, \qquad \underline{E}_2 = \begin{pmatrix} 1 \\ \frac{1-i\sqrt{39}}{4} \end{pmatrix}.$$

The Jordan form of the matrix M_2 is

$$J = \begin{pmatrix} 6 & \sqrt{39} \\ -\sqrt{39} & 6 \end{pmatrix},$$

and the matrix P that relates J and M_2 is obtained by computing the vectors

$$\underline{e}_1 = \mathrm{Re}(\underline{E}_1) = \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix}, \qquad \underline{e}_2 = \mathrm{Im}(\underline{E}_1) = \begin{pmatrix} 0 \\ \frac{\sqrt{39}}{4} \end{pmatrix},$$

$$P = (\underline{e}_1, \underline{e}_2) = \begin{pmatrix} 1 & 0 \\ \frac{1}{4} & \frac{\sqrt{39}}{4} \end{pmatrix}.$$

(c) Finally, we do the same analysis for matrix M_3,

$$M_3 = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

The characteristic polynomial is:

$$(3 - \lambda)(1 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = 0,$$

which gives eigenvalues $\lambda_1 = \lambda_2 = 2$. We therefore have a single eigenvector to compute

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix},$$

which gives $3a + b = 2a$ and $-a + b = 2b$. A solution to these equations is $a = 1$ and $b = -1$. The eigenvector is

$$\underline{E}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The Jordan form of M_3 is

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},$$

and in order to obtain the matrix P we need to find one more independent vector, the Jordan vector $\underline{J}_1$, by solving the equation

$$(M_3 - 2I)\underline{J}_1 = \underline{E}_1.$$

In matrix form,

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which gives $a+b=1$ and $-a-b=-1$. A solution to this equation is to take $a=0$ and $b=1$. Therefore

$$\underline{J}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix P is given by

$$P = (\underline{E}_1, \underline{J}_1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

2. We now need to solve the system of equations

$$\dot{\underline{x}} = A\underline{x},$$

where A is each of the matrices of exercise 1 and $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(a) Let us start with matrix M_1. In matrix form, the equations would be

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 13 & 9 \\ -18 & -14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Because the eigenvalues of M_1 are real and of different signs, the fixed point at the origin is a saddle. The best way of solving the equations is to solve the system first in its canonical form, that is the equations

$$\dot{\underline{y}} = J\underline{y},$$

where J is the Jordan form of M_1 obtained before. This system of equations is nothing but,

$$\dot{y}_1 = 4y_1, \qquad \dot{y}_2 = -5y_2,$$

whose general solutions are $y_1 = C_1 e^{4t}$ and $y_2 = C_2 e^{-5t}$, where C_1 and C_2 are arbitrary constants. In vector form this means that

$$\underline{y} = C_1 e^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is easy to obtain the solutions x_1 and x_2 from this. We just have to multiply the solution above by the matrix P. This gives

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P\underline{y} = C_1 e^{4t} P \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-5t} P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 e^{4t}\underline{E}_1 + C_2 e^{-5t}\underline{E}_2,$$

or, in components

$$x_1 = C_1 e^{4t} + C_2 e^{-5t}, \qquad x_2 = -C_1 e^{4t} - 2C_2 e^{-5t}.$$

The phase space diagrams, both in the "canonical" coordinates $y_1 - y_2$ and the original variables $x_1 - x_2$ are given in figure 1.

(b) For M_2 we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -10 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Because the eigenvalues of M_2 are complex conjugated to each other, with positive real part, the fixed point at the origin would be an unstable focus.
In order to find the solution to the equations we first solve the canonical system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 6 & \sqrt{39} \\ -\sqrt{39} & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We saw in the lecture that this type of system is always solved by using the change of variables $y_1 = r\cos\theta$, $y_2 = r\sin\theta$. Then, the equations are solved to

$$r = r_0 e^{6t}, \qquad \theta = -\sqrt{39}t + \theta_0,$$

where r_0 and θ_0 are arbitrary constants. So the solutions are

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = r_0 e^{6t}\cos(-\sqrt{39}t+\theta_0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{6t}\sin(-\sqrt{39}t+\theta_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To obtain the solutions for x_1, x_2 we just have to multiply the vector above by the matrix P,

$$\underline{x} = P\underline{y} = r_0 e^{6t}\cos(-\sqrt{39}t+\theta_0)P\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{6t}\sin(-\sqrt{39}t+\theta_0)P\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= r_0 e^{6t}\cos(-\sqrt{39}t+\theta_0)\underline{e}_1 + r_0 e^{6t}\sin(-\sqrt{39}t+\theta_0)\underline{e}_2,$$

and recalling the expressions for the vectors $\underline{e}_1, \underline{e}_2$ we get

$$x_1 = r_0 e^{6t}\cos(-\sqrt{39}t+\theta_0),$$

$$x_2 = \frac{1}{4}r_0 e^{6t}\cos(-\sqrt{39}t+\theta_0) + \frac{\sqrt{39}}{4}r_0 e^{6t}\sin(-\sqrt{39}t+\theta_0).$$

The phase space diagrams, both in the $y_1 - y_2$ and $x_1 - x_2$ variables are given in figure 2.

(c) For M_3 we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Because the eigenvalues of M_3 are real, positive and equal to each other, the fixed point would be an unstable improper node. The solution to the system $\dot{\underline{y}} = J\underline{y}$ in this case can be obtained by looking at the equation in components:

$$\dot{y}_1 = 2y_1 + y_2, \qquad \dot{y}_2 = 2y_2.$$

The second equation can be solved directly to $y_2 = C_2 e^{2t}$. Substituting into the first equation and using an integrating factor, one finds $y_1 = C_2 e^{2t} t + C_1 e^{2t}$. In vector form we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (C_1 + C_2 t)e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From this we get also

$$\underline{x} = P\underline{y} = (C_1 + C_2 t)e^{2t}P\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t}P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (C_1 + C_2 t)e^{2t}\underline{E}_1 + C_2 e^{2t}\underline{J}_1.$$

In components this gives

$$x_1 = (C_1 + C_2 t)e^{2t}, \qquad x_2 = -(C_1 + C_2 t)e^{2t} + C_2 e^{2t}.$$

The phase space diagrams are given in figure 3.

3.

$$(a) \quad \dot{x}_1 = -17x_1 + 39x_2 + 13, \qquad \dot{x}_2 = -6x_1 + 13x_2 + 26.$$

In matrix form this is

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{\underline{x}}} = \underbrace{\begin{pmatrix} -17 & 39 \\ -6 & 13 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\underline{x}} + \underbrace{\begin{pmatrix} 13 \\ 26 \end{pmatrix}}_{\underline{b}}.$$

The fixed point of the system is the solution of $A\underline{x} + \underline{b} = 0$, that is

$$\underline{a} = -A^{-1}\underline{b} = -\begin{pmatrix} 1 & -3 \\ \frac{6}{13} & -\frac{17}{13} \end{pmatrix}\begin{pmatrix} 13 \\ 26 \end{pmatrix} = \begin{pmatrix} 65 \\ 28 \end{pmatrix}.$$

One can rewrite the system of equations above in the form $\dot{\underline{z}} = A\underline{z}$ by defining the new vector $\underline{z} = \underline{x} - \underline{a}$. We will now solve the equations $\dot{\underline{z}} = A\underline{z}$ by rewriting this equation in the canonical form

$$\dot{\underline{y}} = J\underline{y}, \quad \text{with} \underline{z} = P\underline{y},$$

and $J = P^{-1}AP$. We now construct the matrix P in the usual form.
The characteristic polynomial A is $\lambda^2 + 4\lambda + 13$ and it has zeroes at $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$. These are the eigenvalues of A. The eigenvectors can be computed to

$$\underline{E}_1 = \begin{pmatrix} \frac{5-i}{2} \\ 1 \end{pmatrix}, \qquad \underline{E}_2 = \begin{pmatrix} \frac{5+i}{2} \\ 1 \end{pmatrix}.$$

The matrix P is built out of the real and imaginary parts of the eigenvector $\underline{E}_1$ as

$$P = (\underline{e}_1, \underline{e}_2) = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

Given the form of the eigenvalues we can already say that the fixed point of the system is an stable focus. This also means that the solutions for y_1, y_2 are given by

$$y_1 = r_0 e^{-2t}\cos(-3t + \theta_0), \qquad y_2 = r_0 e^{-2t}\sin(-3t + \theta_0),$$

with r_0, θ_0 arbitrary constants. In vector form this is

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = r_0 e^{-2t}\cos(-3t + \theta_0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{-2t}\sin(-3t + \theta_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore the solution for $\underline{z}$ is,

$$\begin{aligned} \underline{z} = P\underline{y} &= r_0 e^{-2t}\cos(-3t + \theta_0)P\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{-2t}\sin(-3t + \theta_0)P\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= r_0 e^{-2t}\cos(-3t + \theta_0)\underline{e}_1 + r_0 e^{-2t}\sin(-3t + \theta_0)\underline{e}_2, \end{aligned}$$

or, in components

$$z_1 = \frac{5}{2}r_0 e^{-2t}\cos(-3t + \theta_0) - \frac{1}{2}r_0 e^{-2t}\sin(-3t + \theta_0), \quad z_2 = r_0 e^{-2t}\cos(-3t + \theta_0).$$

Finally, the solution for $\underline{x}$ is just obtained by adding the fixed point to the solution for $\underline{z}$, that is

$$\underline{x} = \underline{a} + r_0 e^{-2t}\cos(-3t + \theta_0)\underline{e}_1 + r_0 e^{-2t}\sin(-3t + \theta_0)\underline{e}_2,$$

or, in components

$$x_1 = 65 + \frac{5}{2}r_0 e^{-2t}\cos(-3t + \theta_0) - \frac{1}{2}r_0 e^{-2t}\sin(-3t + \theta_0), \quad x_2 = 28 + r_0 e^{-2t}\cos(-3t + \theta_0).$$

The phase space diagrams in the $y_1 - y_2$ and $x_1 - x_2$ planes are shown in figure 4.
We will now repeat the same sort of analysis for the system of equations

$$(b) \quad \dot{x}_1 = 6x_2 + 6, \qquad \dot{x}_2 = -x_1 + 5x_2 + 1.$$

In matrix form this is

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{\underline{x}}} = \underbrace{\begin{pmatrix} 0 & 6 \\ -1 & 5 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\underline{x}} + \underbrace{\begin{pmatrix} 6 \\ 1 \end{pmatrix}}_{\underline{b}}.$$

The fixed point of the system is the solution of $A\underline{x} + \underline{b} = 0$, that is

$$\underline{a} = -A^{-1}\underline{b} = -\begin{pmatrix} \frac{5}{6} & -1 \\ \frac{1}{6} & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}.$$

One can rewrite the system of equations above in the form $\dot{\underline{z}} = A\underline{z}$ by defining the new vector $\underline{z} = \underline{x} - \underline{a}$. We will now solve the equations $\dot{\underline{z}} = A\underline{z}$ by rewriting this equation in the canonical form

$$\dot{\underline{y}} = J\underline{y}, \quad \text{with}\underline{z} = P\underline{y},$$

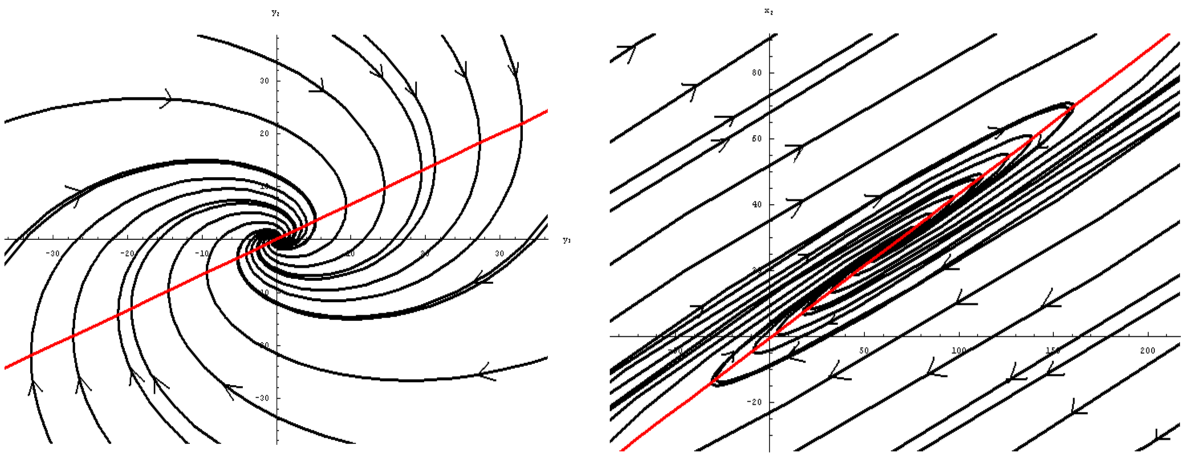


Figure 1: In both diagrams the trajectories flow towards the fixed point. In both cases they do so clockwise. This can be checked by looking at $\dot{y}_1$ at $y_1 = 0$. In that case $\dot{y}_1 = 3y_2$ which is positive if y_2 is positive. So the trajectories must cross the positive y_2 axis in the direction of increasing y_1 (that is the clockwise direction). A similar analysis for $\dot{x}_1$ shows that the trajectories move clockwise in the second diagram as well. The second feature that characterizes fixed points of focus type is the slope of the trajectories. One can find the line where the trajectories become vertical by solving the equation $\dot{y}_1 = 0$, that is $y_2 = 2y_1/3$ or $\dot{x}_1 = 0$, that is $x_2 = (17x_1 - 13)/39$. This gives the equations of two red lines shown in the graphs. Notice that now in the second graph the trajectories flow to the fixed point at $x_1 = 65, x_2 = 28$.

and $J = P^{-1}AP$. We now construct the matrix P in the usual form.
The characteristic polynomial A is $\lambda^2 - 5\lambda + 6$ and it has zeroes at $\lambda_1 = 3$ and $\lambda_2 = 2$. These are the eigenvalues of A. The eigenvectors can be computed to

$$\underline{E}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \underline{E}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The matrix P is built as

$$P = (\underline{E}_1, \underline{E}_2) = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}.$$

Given the form of the eigenvalues we can already say that the fixed point of the system is an unstable node. This also means that the solutions for y_1, y_2 are given by

$$y_1 = C_1 e^{3t}, \qquad y_2 = C_2 e^{2t},$$

with C_1, C_2 arbitrary constants. In vector form this is

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore the solution for $\underline{z}$ is,

$$\underline{z} = P\underline{y} = C_1 e^{3t} P \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 e^{3t} \underline{E}_1 + C_2 e^{2t} \underline{E}_2,$$

or, in components

$$z_1 = 2C_1 e^{3t} + 3C_2 e^{2t}, \quad z_2 = C_1 e^{3t} + C_2 e^{2t}.$$

Finally, the solution for $\underline{x}$ is just obtained by adding the fixed point to the solution for $\underline{z}$, that is

$$x_1 = -4 + 2C_1 e^{3t} + 3C_2 e^{2t}, \qquad x_2 = -1 + C_1 e^{3t} + C_2 e^{2t}.$$

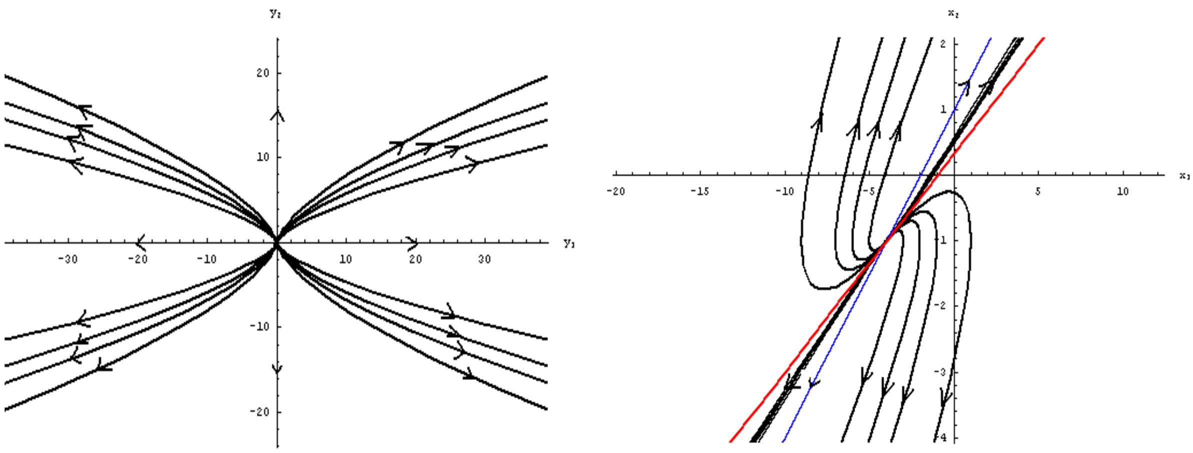


Figure 2: In both diagrams the trajectories flow away from the fixed point. In the $y_1 - y_2$ diagram, trajectories tend to become vertical at the origin and horizontal very far away from the origin. In other words, trajectories take the direction of the y_2-axis near the origin and the direction of the y_1 axis away from the origin. The role of these two directions is replaced by the directions of the eigenvectors $\underline{E}_1, \underline{E}_2$ in the second diagram. The $\underline{E}_1$ direction is indicated in blue, whereas the $\underline{E}_2$ direction is indicated in red. In fact these are not exactly the directions of the eigenvectors. They are obtained by drawing those and then shifting the origin from $(0, 0)$ to (-4,-1).

The phase space diagrams in the $y_1 - y_2$ and $x_1 - x_2$ planes are shown in figure 5. Finally, let us look at the equations,

$$(c) \quad \dot{x}_1 = -4x_1 + 9x_2 + 2, \qquad \dot{x}_2 = -x_1 + 2x_2 + 1,$$

In matrix form this is

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{\underline{x}}} = \underbrace{\begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\underline{x}} + \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\underline{b}}.$$

The fixed point of the system is the solution of $A\underline{x} + \underline{b} = 0$, that is

$$\underline{a} = -A^{-1}\underline{b} = -\begin{pmatrix} 2 & -9 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

One can rewrite the system of equations above in the form $\dot{\underline{z}} = A\underline{z}$ by defining the new vector $\underline{z} = \underline{x} - \underline{a}$. We will now solve the equations $\dot{\underline{z}} = A\underline{z}$ by rewriting this equation in the canonical form

$$\dot{\underline{y}} = J\underline{y}, \quad \text{with} \underline{z} = P\underline{y},$$

and $J = P^{-1}AP$. We now construct the matrix P in the usual form.
The characteristic polynomial A is $\lambda^2 + 2\lambda + 1$ and it has zeroes at $\lambda_1 = -1$ and $\lambda_2 = -1$. These are the eigenvalues of A. There is therefore only one eigenvector which can be computed to

$$\underline{E}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

In order to find the matrix P we need to construct one further vector. We called this vector the Jordan vector $\underline{J}_1$. It is obtained by solving the equation $(A + I)\underline{J}_1 = \underline{E}_1$. A solution to this equation is

$$\underline{J}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The matrix P is built as

$$P = (\underline{E}_1, \underline{J}_1) = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

Given the form of the eigenvalues we can already say that the fixed point of the system is an stable improper node. This also means that the solutions for y_1, y_2 are given by

$$y_1 = (C_1 + C_2 t)e^{-t}, \qquad y_2 = C_2 e^{-t},$$

with C_1, C_2 arbitrary constants. In vector form this is

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (C_1 + C_2 t)e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore the solution for $\underline{z}$ is,

$$\underline{z} = P\underline{y} = (C_1 + C_2 t)e^{-t} P \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (C_1 + C_2 t)e^{-t}\underline{E}_1 + C_2 e^{-t}\underline{J}_1,$$

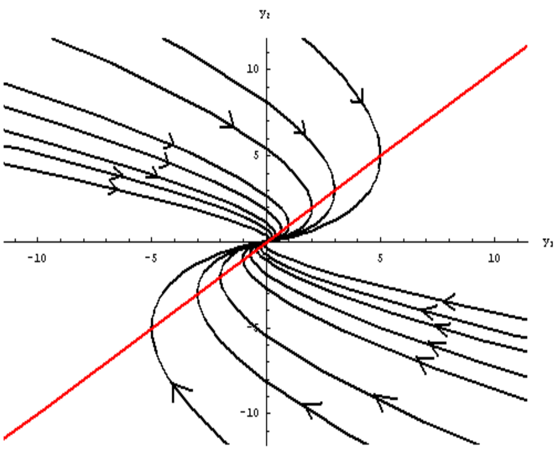
or, in components

$$z_1 = 3(C_1 + C_2 t)e^{-t} + 2C_2 e^{-t}, \quad z_2 = (C_1 + C_2 t)e^{-t} + C_2 e^{-t}.$$

Finally, the solution for $\underline{x}$ is just obtained by adding the fixed point to the solution for $\underline{z}$, that is

$$x_1 = 5 + 3(C_1 + C_2 t)e^{-t} + 2C_2 e^{-t}, \qquad x_2 = 2 + (C_1 + C_2 t)e^{-t} + C_2 e^{-t}.$$

The phase space diagrams in the $y_1 - y_2$ and $x_1 - x_2$ planes are shown in figure 6.



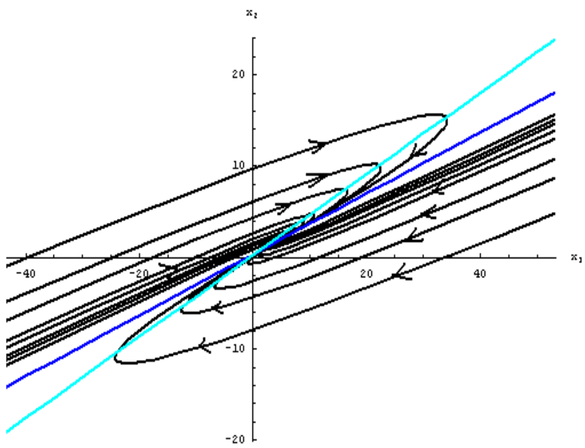


Figure 3: In both diagrams the trajectories flow towards the fixed point. In the $y_1 - y_2$ diagram, trajectories tend to become vertical horizontal at the origin. In other words, trajectories take the direction of the y_1-axis near the origin. The role of the y_1-axis is replaced by the directions of the eigenvector $\underline{E}_1$ in the second diagram (the dark blue line). In both diagrams there is a place where all trajectories become vertical (infinite slope). This corresponds to the red line in the 1st diagram and the light blue line in the second diagram. The red line is the place where $\dot{y}_1 = 0$, that is $y_2 = y_1$. The light blue line is the place where $\dot{x}_1 = 0$, that is $-4x_1 + 9x_2 + 2 = 0$. Notice that the eigenvector $\underline{E}_1$ is a vector that starts at the origin and ends at the point $(3, 1)$. This defines the direction of the dark blue line. However, because in the second diagram the origin has been moved to the point $(5, 2)$, the eigenvector also has to be shifted there.

4. *Consider the equation* $\dot{\underline{x}} = A\underline{x}$ *with*

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad x \in [0, \pi].$$

- *Show that the fixed point at the origin is simple.*
  A simple fixed point in one for which the matrix A has $\det(A) \neq 0$. The determinant of A in this case is actually 1 for all values of θ, so all fixed points are simple in this case.
- *Calculate the eigenvalues of* A. *Hence determine the values of* θ *for which the origin is:*
  The eigenvalues of A are the solutions of the equation:

$$(\lambda - \cos\theta)^2 + (\sin\theta)^2 = \lambda^2 + (\cos\theta)^2 - 2\lambda\cos\theta + (\sin\theta)^2 = \lambda^2 - 2\lambda\cos\theta + 1 = 0,$$

  that is $\lambda = \frac{2\cos\theta \pm \sqrt{4(\cos\theta)^2 - 4}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$.

  (a) *a stable star node*: a star node only happens when A is a diagonal matrix, proportional to the identity. This occurs whenever $\sin\theta = 0$, that is for $\theta = 0, \pi$. For the node to be stable we need $\cos\theta > 0$, so this selects the value of θ to be only $\theta = 0$.

  (b) *an unstable star node:* this is as the previous case, but now we need $\cos\theta < 0$ which selects out the other values of θ, that is $\theta = \pi$.

  (c) *a centre*: this is when the eigenvalues are complex conjugated to each other with vanishing real part. That is $\cos\theta = 0$, which corresponds to $\theta = \frac{\pi}{2}$.

  (d) *a stable focus*: this requires complex conjugated eigenvalues, with negative real part. This means that $\cos\theta < 0$ with $\sin\theta \neq 0$. This corresponds to the interval $\frac{\pi}{2} < \theta < \pi$.

  (e) *an unstable focus*: this requires complex conjugated eigenvalues, with positive real part. This means that $\cos\theta > 0$ with $\sin\theta \neq 0$. This corresponds to the interval $0 < \theta < \frac{\pi}{2}$.