Worksheet 3

Introduction

On this sheet we will begin investigating some of the simplest forms of equations, those corresponding to straight lines. We will use DERIVE to determine the intersection of a straight line with a given curve (which may also be a straight line) in the plane.

Problem 4

- Use DERIVE to Author the equation $y = 2x + 4$. Hence plot the curve.

  From the graph write down the coordinates where

  (i) the line cuts the y-axis, $(0, \ldots\ldots\ldots)$ (the y-*intercept*).

  (ii) the line cuts the x-axis, $(\ldots\ldots\ldots, 0)$ (the x-*intercept*).

- Repeat the above with the equation $y = -2x + 4$.

  (i) the line cuts the y-axis, $(0, \ldots\ldots\ldots)$ (the y-*intercept*).

  (ii) the line cuts the x-axis, $(\ldots\ldots\ldots, 0)$ (the x-*intercept*).

- Using the graph write down the slopes of the two lines:

  (i) Slope of $y = 2x + 4$ is ....

  (ii) Slope of $y = -2x + 4$ is ....

This should remind you of some general theory about straight lines. The equation of a straight line is often given by $y = mx + c$ where m is the *gradient* and c is the y-*intercept*. The value of c is determined by putting $x = 0$ and solving for y.

The *gradient of a straight line through two points* (x_1, y_1) *and* (x_2, y_2) is, by definition,

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{provided that } x_1 \neq x_2.$$

From this we obtain that the *equation of a straight line through the two points* is given by

$$y - y_1 = m(x - x_1).$$

The *most general* equation for a straight line is given by

$$ax + by = c.$$

This allows lines of the form $ax = c$, i.e., lines with infinite gradient.

Problem 5

- Write down the equations of the three straight lines through pairs of the following points A, B, and C:

$$A = (-3, -7), \qquad B = (2, 6), \qquad C = (2, -4).$$

....

- Plot these three lines with DERIVE and check the lines do pass through the three points. Then use your diagram to deduce the area of the enclosed triangle.

Problem 6

Start with a fresh plot window (or clear an existing one by using <Ctrl D>).

- Plot the straight lines

$$\text{(i)}\quad 2y+3x=2 \qquad \text{(ii)}\quad 2y+3x=4 \qquad \text{(iii)}\quad 2y+3x=6 \qquad \text{(iv)}\quad 2y+3x=8.$$

- From your graph describe what happens to the straight lines as the right hand side of each equation increases.

....

- On your diagram plot the curve $x^2+y^2=1$.

- By experimentation (i.e., by plotting $2y+3x=c$ for different values of c) find the largest value of c (to 1 decimal place) such that $2y+3x=c$ and $x^2+y^2=1$ have at least one point in common.

$c =$ ....

The straight line that satisfies the above requirement is a tangent to the curve. To solve this problem algebraically rather than approximately using diagrams we need to find the value of c such that the line and the curve intersect in a single point rather than two points (compare with your diagram above).

- From $2y+3x=c$ deduce that $y=\frac{c}{2}-\frac{3}{2}x$. Use this to substitute for y in the equation $x^2+y^2=1$ and hence show that

$$13x^2-6xc+(c^2-4)=0. \tag{1}$$

- Treating equation (1) as a quadratic in x show that the condition for a single root is $c^2=13$. Hence find the values of c and the coordinates of the point of contact of the straight line and the curve.

$c =$ .... point = (......... , ...).

Note that this gives two possible values of c: plot both straight lines. The values of c obtained are the maximum and minimum values of c for which the straight line and the curve intersect only once.

Problem 7

A company produces 2 products X and Y. Denoting the production levels by x and y respectively, measured in thousands of units produced, a constraint on the production plant is given by $2x^2+y^2\leq 1$. If the profit on the production of 1000 of X is £3000 and the profit on the production of 1000 of Y is £4000 show that the total profit in units of £1000 is given by $P=4y+3x$. This problem asks you to find the production levels to produce maximum profit and the corresponding amount of profit generated.

- This problem is mathematically the similar to Problem 6.

- To get an idea of the solution use DERIVE to draw the curve $2x^2+y^2=1$.

  Note that production can only take place either on or inside this curve with x and y non-negative.

- Plot the 2 straight lines $P=4y+3x$ with $P=4$ and $P=5$.

- Write down a rough estimate of the maximum profit and the production levels.

profit = ... levels: $x =$ .... and $y =$ ....

- Proceed as in the second part of Problem 6 to obtain accurate values for the maximum profit and production levels.

profit = ... levels: $x = ...$ and $y = ...$

Problem 8

- Plot the following straight lines:

  (i) $x = 0$ (ii) $y = 0$ (iii) $3y + 2x = 6$ (iv) $4x + 4y = 9$ (v) $y + 2x = 4$.

- The boundary and interior of the pentagon bounded by these lines is specified by the inequalities

  (i) $x \geq 0$ (ii) $y \geq 0$ (iii) $3y + 2x \leq 6$ (iv) $4x + 4y \leq 9$ (v) $y + 2x \leq 4$.

Use the trial method of the above 2 problems to find the maximum value of P satisfying these inequalities where P is given by $P = 2y + 3x$. Specify the values of x and y at this point.

$x = ...$ and $y = ...$