Worksheet 6

Problem 18

On the last sheet we saw that a set of simultaneous equations could be written in the matrix form $A\mathbf{x} = \mathbf{b}$, and that the solution was given by $\mathbf{x} = A^{-1}\mathbf{b}$ (provided that we could make sense of A^{-1}). Before moving on we note that this idea can also be applied to larger sets of equations and DERIVE can provide a solution in the same way as before.

Given the set of equations

$$\begin{array}{rrrcr} x & +y & +3z & = & 6 \\ 2x & +y & +z & = & -6 \\ x & -y & -z & = & 12 \end{array}$$

we can rewrite this as

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 12 \end{pmatrix}$$

that is to say in the form $A\mathbf{x} = \mathbf{b}$.

- For this example write down A, $\mathbf{x}$, and $\mathbf{b}$.
- Apply the method of problem 16 using DERIVE to solve the equations.

Note that the only change is that A now has 3 rows and 3 columns and $\mathbf{b}$ has 3 rows and 1 column.

Problem 19

Write down the following equations in the form $A\mathbf{x} = \mathbf{b}$ and hence use DERIVE to obtain their solutions.

(a)
$$\begin{array}{rrrl} x & +4y & -2z & = 4, \\ x & -y & +z & = 0, \\ 2x & -3y & +z & = 4. \end{array}$$

(b)
$$\begin{array}{rrrrl} x & -7y & +2z & -w & = 14, \\ x & -3y & +z & +w & = 42, \\ 2x & -y & +z & -w & = 56, \\ -x & +y & -2z & +3w & = 14. \end{array}$$

(c)
$$\begin{array}{rrrrl} x & -7y & +2z & -w & = 1, \\ & 3y & +z & +w & = 1, \\ & & 4z & -w & = 1, \\ & & & 3w & = 1. \end{array}$$

The matrix in (c) is said to be in *upper triangular form*.

Matrix multiplication I

On sheet 5 we rewrote $\begin{array}{rr} 2x & +3y \\ 4x & -3y \end{array}$ in the matrix form $\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. That is, we have implicitly written these equations in matrix form as

$$\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 4x - 3y \end{pmatrix}.$$

In this way we have established a rule for multiplying a 2×2 matrix by a 2×1 matrix. More precisely, we see that we are "multiplying" (in a rather strange way) each row of the first matrix with the column of the second matrix.

Problem 20

Use this rule to calculate:

(a) $\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

The product of a row with a column

The product of a row with a column is formed by summing the products of corresponding terms of the row and column. For example

$$\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 1 \times 4 + 2 \times 3 = 4 + 6 = 10.$$

NB: This definition assumes that the number of elements in the row is the same as the number of elements in the column.

Problem 21

Where possible multiply together the following:

(a) $\begin{pmatrix} 1 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

(e) $\begin{pmatrix} -1 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Why is it not possible to multiply together the row and column in (f)?

Matrix multiplication II

We can now define the product of two matrices by considering the first matrix as a series of rows, and the second as a series of columns. For example, consider

$$\begin{pmatrix} 2 & 3 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}.$$

We can form the four row-column products and place them in a new matrix:

$$\begin{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} & \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ \begin{pmatrix} -1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} & \begin{pmatrix} -1 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 14 & 19 \\ -13 & -17 \end{pmatrix}.$$

Note that the order is important! Products in the first row of our new matrix use the first row of the left-hand matrix, etc.

Problem 22

For each of the following products compute AB *by hand*:

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}$. (b) $A = \begin{pmatrix} 0 & 3 \\ -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 \\ 4 & 4 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (d) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(e) $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$. (f) $A = \begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

From (a) and (f) can you deduce a general result about matrix products?

Problem 23

We will now see how to use DERIVE to multiply matrices:

- **Author** $\rightarrow$ **Matrix**, then change to 2 rows and 2 columns **OK**.
- Enter the matrix values for A in Problem 22(a) **OK**.
- Repeat and enter the matrix values for B in Problem 22(a).

Suppose that the matrix A is now on line $\#1$ and B is now on line $\#2$. We can now compute the product AB as follows:

- Author the expression $\#1.\#2$ (Note: you *must* include the dot!)
- **Simplify** $\rightarrow$ **Basic** **OK**.

Use this method to check all your answers in Problem 22.

Problem 24

State which of the following products are possible and which are not, giving reasons in the latter case. Compute by hand those products that are possible and check your answer using DERIVE.

(a) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 & 1 \\ -1 & 2 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 3 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$,

(e) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

Inverse and identity matrices

Given a matrix A, we define the *inverse* of A (if it exists) to be a matrix A^{-1} such that

$$A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 *identity matrix*.

We have already used the notation A^{-1} on the previous problem sheet when we worked out the solution to a system of linear equations. This is not accidental; we will now explain how the inverse matrix can be used to solve systems of equations.

Consider the equation $A\mathbf{x} = \mathbf{b}$ in the case

$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{1}$$

It can be checked that the inverse of A is the matrix

$$A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

(i.e., that the product $A^{-1}A$ equals the identity matrix I). Suppose that we multiply both sides of equation (1) by A^{-1}:

$$\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Simplifying we see that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This last equation can be written as $\mathbf{x} = A^{-1}\mathbf{b}$, which is the formula that we gave on the last sheet for solving the system of equations. Thus we see that the inverse matrix, when it exists, allows us to solve systems of linear equations.

Of course, we can also define inverse matrices for $n \times n$ matrices provided we take I to be a suitably sized identity matrix, and use them to solve larger systems of equations.

Problem 25

- Use DERIVE to compute the inverse of the following matrices and hence check that $A^{-1}A = I$, where I is a suitably sized identity matrix. To do this use DERIVE to enter the matrix in the usual fashion. If this matrix is in line $\#1$, then **Author** the expression #1^(-1) and **Simplify**. This will give the inverse if it exists.
- If the inverse is displayed on line $\#2$, then use DERIVE to compute the product $\#1.\#2$ (Don't forget the .).

(a) $\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$, (d) $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$,

(e) $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 0 & 2 & 4 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, (g) $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 3 & 2 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$.