Worksheet 7

Introduction

So far on these worksheets we have concentrated on equations for which it is possible to calculate an exact solution. However, in most cases, the solution of the equation $f(x) = 0$ cannot be computed exactly, but instead requires the use of some numerical methods. The general approach is to try and find some approximate solution (or solutions) by, for example, plotting $y = f(x)$ and then looking to see where the curve cuts the x-axis. In this sheet we investigate *iterative* or *fixed point* methods for solving $f(x) = 0$.

Problem 26

- Use DERIVE to plot the curve $y = x - 2\sin(x)$ and from your graph write down approximations to the solution of $x - 2\sin(x) = 0$.

....

An alternative way of approaching this problem is to plot the two curves $y = x$ and $y = 2\sin(x)$, and then look at the graph to see where the two curves intersect.

- Use DERIVE to plot the curve $y = x$ and $y = 2\sin(x)$, and hence write down the values of x where the curves intersect. Check that these answers are the same as those above.

....

If we write $f(x) = x - 2\sin(x)$ then in the last problem we have seen two ways of writing out our equation:

$$\begin{array}{ccc} x - 2\sin(x) = 0 & \text{or} & x = 2\sin(x) \\ f(x) = 0 & & x = g(x) \end{array}$$

where we have introduced the new function $g(x) = 2\sin(x)$.

Problem 27

- Solve the quadratic $16x^2 - 16x + 3 = 0$ to give two solutions $x = \alpha$ and $x = \beta$.

$\alpha =$ .... $\beta =$ ....

- Write the quadratic in the form $x = g(x)$.

....

- Show by substitution that $\alpha = g(\alpha)$ and $\beta = g(\beta)$.

For any function g, we call a value $x = a$ such that $a = g(a)$ a *fixed point* of $g(x)$. If $f(x) = 0$ is rearranged to give $g(x) = x$ then the fixed points of g are precisely the solutions of f.

We will now return to the function considered in Problem 26. So far the only method we have to find a solution is to draw the graph and estimate the answer. In the next problem we will see how to estimate the answer numerically using what is called an *iterative process*.

Problem 28

From the graphs that you plotted in Problem 26 we can see that $x = 2$ is an approximate solution to $x = 2\sin(x)$. We set $x_1 = 2$.

- Calculate $2\sin(2)$ to two decimal places, and denote this value by x_2. That is, $x_2 = 2\sin(x_1)$. (Note that the angle is in radians!)

  Since $x_2 \neq x_1$, i.e. $2 \neq 2\sin(2)$, we see that x_1 is not a solution to our problem.

- Calculate $2\sin(x_2)$ to two decimal places, and denote this value by x_3.

Since $x_3 \neq x_2$, i.e., $x_2 \neq 2\sin(x_2)$, we see that x_2 is not a solution to our problem. However, if you repeat this process (always working to two decimal places) you will eventually reach a value a such that $a = 2\sin(a)$. You will then have a solution to the problem correct to two decimal places. Carry out this procedure and write down the value of a.

. . . .

The calculations above suggest the following general procedure for solving an equation of the form $f(x) = 0$ to a given level of accuracy:

- Rewrite the equation $f(x) = 0$ in the form $x = g(x)$.
- Start with a first approximation $x = x_1$.
- Iterate using the scheme $x_{i+1} = g(x_i)$ until the two values agree to the required accuracy.

Unfortunately there is no guarantee that this procedure will always work! We will investigate this in the next Problem.

Problem 29

In Problem 27 you probably obtained the rearrangement of the quadratic into the form $x = \frac{16x^2+3}{16}$, and found that the two solutions where $\alpha = \frac{1}{4}$ and $\beta = \frac{3}{4}$.

- Starting with the approximation $x_1 = 0.2$ iterate until you obtain two answers that agree to two decimal places. This should of course give you the exact value for α.
- Starting with $x_1 = 0.8$ carry out four iterations to obtain x_2, x_3, x_4, and x_5. What do you think would happen if you continued to iterate?

. . . .

Carrying out the above procedure by hand can become quite tedious, especially when the number of iterations becomes large. Fortunately DERIVE provides a function which will carry out the iteration for us. To implement this for this problem proceed as follows:

- **Author** $(16x^2 + 3)/16$
- **Author** iterates$(\#1, x, 0.2, 10)$

  This specifies 10 iterations of the expression in line number 1 starting with the value $x = 0.2$. If your expression for $(16x^2 + 3)/16$ is on another line the replace the 1 by the appropriate number.
- To carry out the iterations use: **Simplify** $\rightarrow$ **Approximate** $\rightarrow$ **Approximate**.
- Change the starting value from 0.2 to 0.8 and repeat the exercise.

Your results have shown that the rearrangement of $f(x) = 0$ used above fails to give the second root of the quadratic. This is a very common feature of this method. It can be overcome by using a different rearrangement of $f(x) = 0$.

It is clear that we can rearrange our quadratic into the form $16x^2 = 16x - 3$, which gives $x^2 = \frac{16x-3}{16}$. Hence dividing by x gives $x = \frac{16x-3}{16x} = g(x)$. Thus for the same initial problem we now have a different function $g(x)$ which (we hope) will allow us to find the second root.

- Author the new equation $g(x)$ (check that the expression on your screne is the correct one) and set up the iterates function as above with a starting value of 0.8. This should tend to the solution at $x = \frac{3}{4}$.

- Change the starting value to 0.2 and carry out the iterations. What happens?

. . . .

Problem 30

For each of the following equations use DERIVE to draw the graph $y = f(x)$ and hence obtain estimates for the roots of the equation $f(x) = 0$. Rearrange each equation into the form $x = g(x)$ and hence use the iterates function of DERIVE to obtain the solutions correct to three decimal places.

Note: as above it may be necessary to use more than one rearrangement for each $f(x)$ in order to obtain all of the solutions. You may also need to carry out more than 10 iterations.

(a) $f(x) = x - 2\sin 2x.$

(b) $f(x) = 10x^2 - 10x + 2.$

(c) $f(x) = x - \cos(x).$

(d) $f(x) = x^3 + 2x^2 - 3x - 1.$

(e) $f(x) = 1 - x + \frac{x^2\cos(x)}{2}$ for $0 \leq x \leq 2.$