The millenium problems

3rd year project 2007/08

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- The aim of this project is that you write an introduction explaining what the millenium problems are and then (if you want) concentrate in more detail in one or two of them.
- In order to learn about the various problems I recommend you to buy the book »The millenium problems by Keith Devlin». It can be found in most bookshops and costs around £10. It contains a very nice introduction about each of the problems.

- The millenium problems are 7 mathematical problems which have been selected by a group of experts and declared to be the 7 most difficult open problems in mathematics today. Some of them have been open for more than a century!
- One of the problems (The Poincaré conjecture) was solved last year, but we can still count it for this project.
- An american foundation called the Clay foundation made these problems famous by offering 1million dollars to anyone who would solve one of them.

- As explained in the book, some of the problems are very difficult to describe, even to say what they are about!
- But there are others which are easier to describe (although as difficult to solve). They are:

1. The Riemann hypothesis
2. The Poincaré conjecture
3. The Navier-Stokes equation
4. The P versus NP problem

The Riemann hypothesis

- This is the oldest problem in the list. It goes back to the german mathematician Bernhard Riemann in 1859. Riemann studied a function called the Riemann zeta-function (usually written as $\zeta(s)$).
- His hypothesis is about the zeroes of this function, namely the solutions of the equation $\zeta(s)=0$. There are some solutions to this equation which are called trivial solutions and correspond to all negative integer values of s, s=-1,-2,-3… But there are other zeroes that are non-trivial. Riemann conjectured that the real part of all these zeroes is ½, that is they are of the form s=1/2 + i w for some value of w.
- The Riemann ζ-function is closely related to the prime numbers so that a proof of the conjecture would give very important information about the structure of the primes (many applications!)

The Poincaré conjecture

- This is also a very old problem. It goes back to the french mathematician Henry Poincaré around 1904. The problem has to do with a brach of mathematics called topology. The conjecture has been proven in 2002/03 by G. Perelmann.
- Topology is closely related to geometry. One way of defining it is as a branch of mathematics that tries to clasify geometric objects in groups according to certain properties. For example, let us look at the property that is at the basis of the Poincaré conjecture….

- Let us look at a sphere. This is a 2-dimensional surface. One of the topological properties that characterizes the sphere is that any closed loop drawn on it can be continuously tightened to a point. In topology we would say that any surface having this property is topologically equivalent to an sphere!
- But people in mathematics usually like to generalize things. For example they like to think higher dimensional generalizations of the sphere. Poincaré asked himself the question of whether or not the equivalent of the sphere in higher dimensions will also have a similar property. He stated that this would be the case and that was his conjecture.

The Navier-Stokes equation

- It was introduced by Navier (1822) and later on re-derived by Stokes (1840s). It is a set of equations that describe the dynamics of fluids (that is gases and liquids). Therefore they have wide applications in many areas (e.g. aviation).
- The equations look like

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

http://en.wikipedia.org/wiki/Navier-Stokes_equations#Derivation_and_description

- There are 3 equations, one for each component of the velocity of the fluid (u,v,w).
- The components of the velocity are functions of the space position (x,y,z) and time, t.
- Rho is the fluid density
- p is the pressure, which is also a function of the position and time
- g is the force of gravity.
- mu is a constant which characterizes the viscosity of the fluid.
- The millenium problem in this case is that although very precise numerical solutions to these equations can be obtained there is not yet a mathematical proof of the existence of solutions to these equations in general!

The P versus NP problem

- This is a problem which appears in the context of computer science. In this context one often speaks of P and of E problems.
- P type problems are problems that can be solved in a computer in a time which grows polynomially with the complexity of the problem (if data have size N, the time needed goes as N^k)
- E type problems are problems that can be solved in a computer in a time which grows exponentially with the complexity of the problem (if data have size N, the time needed goes as 2^N)
- Since exponential growth is much faster than polynomial growth E type problems can only be tackled in a computer up to a certain degree of difficulty. Very quickly the time needed for computations is just too long!

- However, this is not all. There are some problems that are neither of one type nor of the other. They are called NP problems (non-deterministic polynomial time problems)
- It just turns out that most important problems in practice are of this type. A famous problem of this type if the «travelling salesman problem».
- They are problems that involve very simple computations but the amount of these computations grows so fast that in the end the seem unmanageable in a computer!
- However there has been a suggestion that there might be a way to transform every NP problem into a P problem! If this was ever proven to be true it could have important implications!