The mathematical theory of Partitions

3rd year project 2010/11

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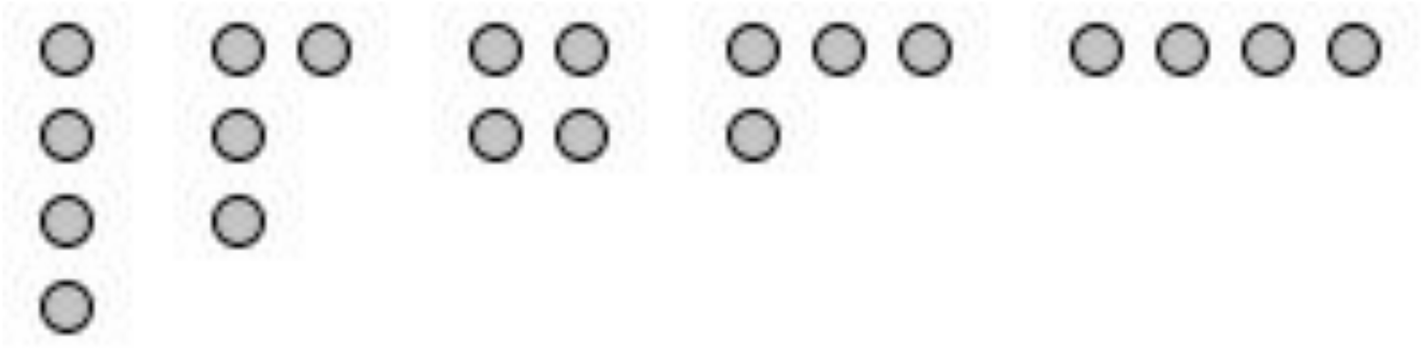
- What is a partition?
- Take a positive integer number, say 5 and write it as a sum of smaller or equal positive integers:

5 = 5
= 4+1
= 3+2
= 3+1+1
= 2+2+1
= 2+1+1+1
= 1+1+1+1+1

We therefore have 7 «ordered» partitions of the number 5

Ordered means that we always start with the biggest number, e.g. we do not count 4+1 and 1+4 as two different partitions

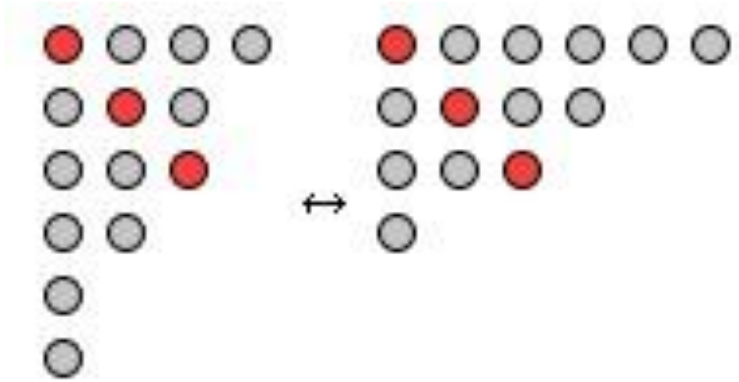
- Each of the sums is a partition of 5. The partition 4+1 is a partition of 5 into two distinct parts. Moreover, this partition has length 2, since it has two parts.
- Partitions can be represented by using diagrams which are called Ferrers diagrams. For example, for the number 4:



1+1+1+1=2+1+1=2+2 = 3+1 =4

http://en.wikipedia.org/wiki/Partition_(number_theory)

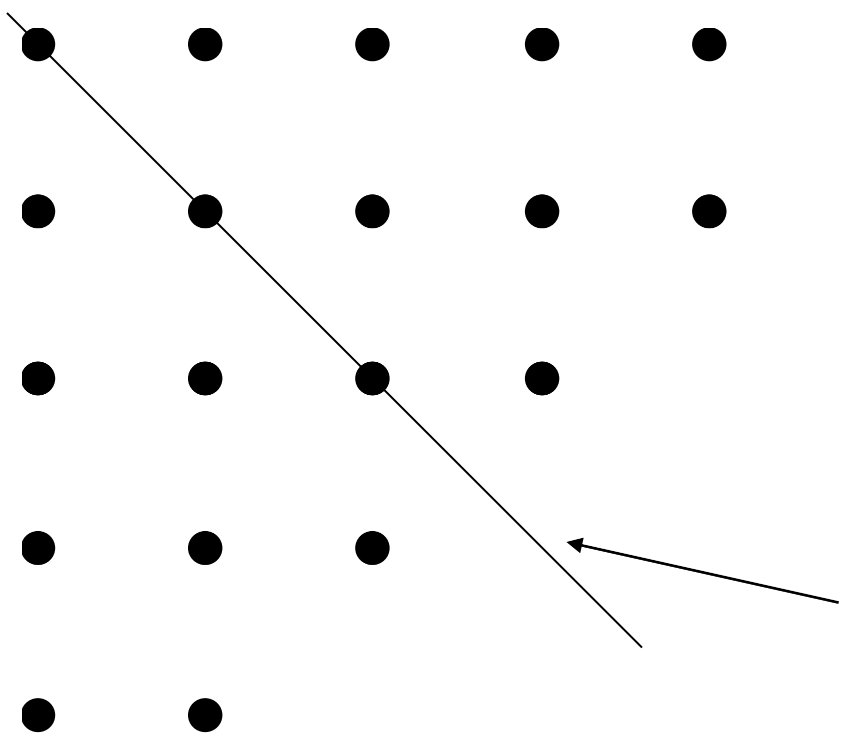
- Given a Ferrers diagram, we define the conjugate Ferrers diagram as the diagram that is obtained by exchanging rows and columns. For example:



4+3+3+2+1+1 6+4+3+1

- The conjugate of a Ferrers diagram has the same number of dots as the original diagram. Therefore, they both represent partitions of the same number. Above it is number 14.

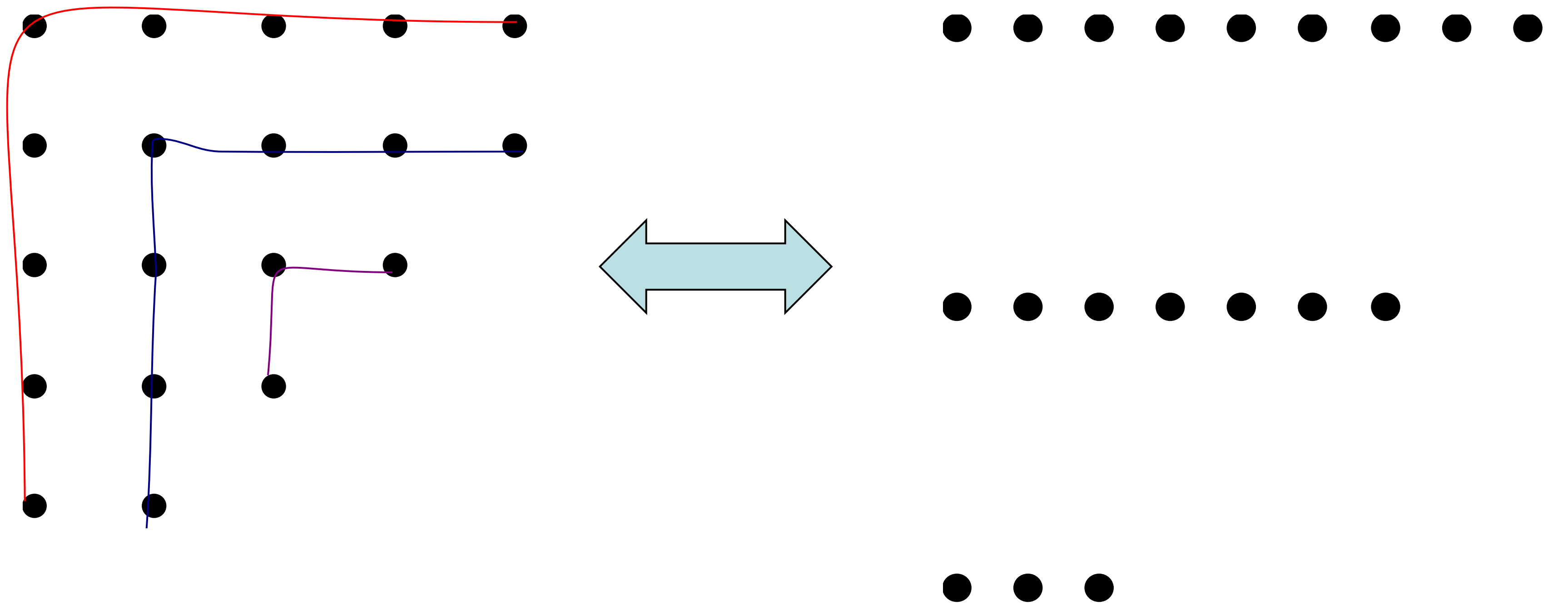
- Some Ferrers diagrams have the property of being identical to their conjugate. In this case they are called self-conjugate Ferrers diagrams. For example:



$$19=5+5+4+3+2$$

- Ferrers diagrams are very useful because they allow to prove many non-trivial relations between partitions. One of the most famous is the following theorem:
- The number of partions of a positive integer into odd distinct parts equals the number of partitions whose Ferrers diagrams are self-conjugate.
- In order to prove this one needs to prove that there is a one-to-one relationship between the two sets. This can be done by using Ferrers diagrams.

- Let us consider a self-conjugate partition such as the one we saw before:



Self-conjugate partition ⟺ Partition into odd distinct parts

- One of the ideas of this project is that you prove several relations between partitions using Ferrers diagrams
- However, there are other ways of studying partitions and one of them is to use generating functions.
- In general, a generating function for a sequence of numbers a_0, a_1, a_2, …. is defined as:

$$G(x) = \sum_{k=0}^{\infty} a_k x^k,$$

- For example, the generating function of non-negative integer numbers {0,1,2…} is

$$G_1(x) = \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

- The last equality follows by using Taylor's expansion

- Euler discovered that one could define generating functions for the number of partitions of a positive integer and that that generating function was given by:

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{k=0}^{\infty} p(k) x^k$$

Number of partitions of the number k

Euler also found the generating function of the number of partions of k into distinct parts

$$\prod_{k=1}^{\infty} (1+x^k) = \sum_{k=0}^{\infty} p_d(k) x^k$$

- It is possible to use generating functions in order to prove some identities between partitions. For example one can show the so-called Euler's parity law : the number of partitions of a number n into distinct parts equals the number of partitions of the same number into odd parts.
- The idea of this project is for you to learn about partitions and carry out several exercises involving what you have learnt.
- You will find some proposed exercises in the photocopies I have given you and you may find other interesting problems by researching the literature.

- The project will then be an introduction to the theory of partitions and generating functions where you use your original solutions to the exercises as examples.
- You may want to use LaTeX for writing your project, in which case you will find information about how to install and use the program here:

http://www.staff.city.ac.uk/o.castro-alvaredo/myprojects.htm

Bibliography

- In the library, you can look for books on «Number Theory». Most such books will have a section about partitions.
- There are also some web-sites about it, including the Wikipedia site:

  http://en.wikipedia.org/wiki/Partition_(number_theory)
- Most web sites will refer to the book « The Theory of Partitions » by George E. Andrews. You can borrow that from me at some point.
- The photocopies from James Tattersall book should be sufficient to do a good project.