CALCULUS 2009: EXAM SOLUTIONS

1. (a) The integration region is the triangle in the xy-plane enclosed by the lines $x = 1$, $y = 0$ and $y = 3x$. $\boxed{3}$

Changing the order of integration we obtain $\boxed{3}$

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=3x} xe^{x^3}\, dy.$$

The integral in y gives $\boxed{2}$

$$\int_{y=0}^{y=3x} xe^{x^3}\, dy = \left[yxe^{x^3}\right]_0^{3x} = 3x^2 e^{x^3}.$$

Plugging this result into the second integral we obtain $\boxed{2}$

$$I = \int_{x=0}^{x=1} 3x^2 e^{x^3}\, dx = \left[e^{x^3}\right]_0^1 = e - 1.$$

The last integral can be easily carried out using the change of variables $t = x^3$.

(b)Since we are computing the volume, we need to carry out an integral of the form

$$V = \int\int\int_R dxdydz,$$

in the region R described by the problem. We also need to use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$, so that the integral becomes

$$V = \int\int\int_R rdrd\theta dz,$$

where R is the same region, described now in terms of cylindrical coordinates. In order to do the problem it is very helpful to get an idea of how the region whose volume we want to evaluate looks. We have two surfaces: a cylinder, defined by the equation $x^2 + y^2 = 2ay$ and a cone defined by the equation $z = 2a - \sqrt{x^2 + y^2}$. We can rewrite the equation of the cylinder as $\boxed{1}$

$$x^2 + (y - a)^2 = a^2.$$

We now see that this a circular cylinder, whose basis is centered at the point $(0, a)$ and has radius a.

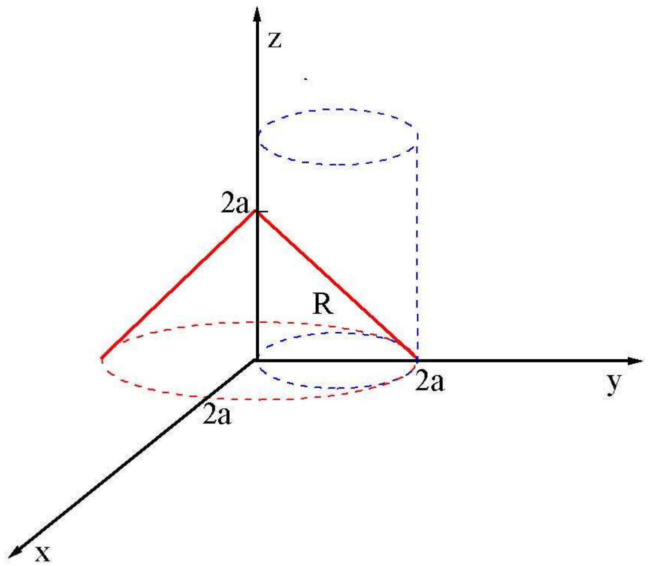
On the other hand if we look at the cone equation, we can first of all write it as:

$$z - 2a = -\sqrt{x^2 + y^2},$$

or

$$(z - 2a)^2 - x^2 - y^2 = 0.$$

The last equation indicates that the cone is centered at the point $(0, 0, 2a)$ in the z-axis and the minus sign in front of the $\sqrt{x^2 + y^2}$ term in the original equation, means that we are choosing the cone sheet that starts at $(0, 0, 2a)$ and extends downwards (the maximum value z can take is $z = 2a$). Combining all this information we can draw the following picture:



and the integration region is the region that is both inside the cylinder and the cone and above the $z = 0$ plane.

We now write the equations of the cylinder and the cone in cylindrical coordinates:

$$x^2 + y^2 = 2ay \qquad \Leftrightarrow \qquad r = 2a\sin\theta,$$

and, looking at the picture of the cylinder above, we see that $0 \leq \theta \leq \pi$. The equation of the cone becomes $z = 2a - r$. Therefore we can write that in the integration region $0 \leq z \leq 2a - r$. Finally we have that $0 \leq r \leq 2a\sin\theta$. 4

Therefore, the integral that we need to compute is:

$$V = \int_{\theta=0}^{\theta=\pi} d\theta \int_{r=0}^{2a\sin\theta} rdr \int_{z=0}^{2a-r} dz.$$

The integration in z is: 2

$$\int_{z=0}^{2a-r} dz = [z]_{z=0}^{z=2a-r} = 2a - r.$$

Putting this into the r-integral we obtain: 1

$$\int_{r=0}^{2a\sin\theta} r(2a-r)dr = \left[ar^2 - \frac{r^3}{3}\right]_{r=0}^{r=2a\sin\theta} = 4a^3\sin^2\theta - 8a^3\sin^3\theta/3 = 4a^3\sin^2\theta(1-2/3\sin\theta).$$

Finally, putting this into the last integral: 2

$$V = 4a^3 \int_{\theta=0}^{\theta=\pi} \sin^2\theta(1-2/3\sin\theta)d\theta = 4a^3\left[\frac{\theta}{2} + \frac{\cos\theta}{2} - \frac{\cos(3\theta)}{18} - \frac{\sin(2\theta)}{4}\right]_{\theta=0}^{\theta=\pi} = 4a^3\left(\frac{\pi}{2} - \frac{8}{9}\right).$$

where we used the identities:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2},$$

and

$$\sin^3\theta = 1/4(3\sin\theta - \sin(3\theta)).$$

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2. (a) First of all we need to find the points at which the first order partial derivatives vanish. These derivatives are 2

$$f_x = (1-2x^2)e^{-x^2+y^2}, \qquad f_y = 2xye^{-x^2+y^2}.$$

Then 1

$$f_x = 0 \quad \Leftrightarrow \quad x = \pm\frac{1}{\sqrt{2}},$$

and 1

$$f_y = 0 \quad \Leftrightarrow \quad x = 0 \;\; \texttt{or} \;\; y = 0.$$

That gives us 2 candidates to be stationary points, that is the points $(\pm 1/\sqrt{2}, 0)$ at which both f_x and f_y vanish. To investigate what type of stationary points this points are, we have to look at the second order partial derivatives: 2

$$f_{xx} = 2x(-3+2x^2)e^{-x^2+y^2}, \qquad f_{yy} = 2x(1+2y^2)e^{-x^2+y^2},$$

$$f_{xy} = f_{yx} = 2y(1-2x^2)e^{-x^2+y^2}.$$

Calling $A = f_{xx}$, $B = f_{xy}$ and $C = f_{yy}$, we find:

i) For the point $(1/\sqrt{2}, 0)$ we have 2

$$A = -2\sqrt{\frac{2}{e}}, \qquad B = 0, \qquad C = \sqrt{\frac{2}{e}}.$$

Then

$$AC - B^2 = -\frac{4}{e} < 0,$$

therefore this point is a saddle point.

ii) For the point $(-1/\sqrt{2}, 0)$ we have 2

$$A = 2\sqrt{\frac{2}{e}}, \qquad B = 0, \qquad C = -\sqrt{\frac{2}{e}}.$$

Then

$$AC - B^2 = -\frac{4}{e} < 0,$$

therefore this point is also a saddle point.

(b) The Taylor expansion of a function of two variables $f(x, y)$ around a point (x_0, y_0) up to second order terms is given by [2]

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0, y_0) = (0, 1)$ and [2]

$$f_x = 2x + y, \quad f_y = 3y^2 + x, \quad f_{xx} = 2, \quad f_{yy} = 6y, \quad f_{xy} = f_{yx} = 1.$$

Therefore [2]

$$f(0,1) = 1, \quad f_x(0,1) = 1, \quad f_y(0,1) = 3, \quad f_{xx}(0,1) = 2, \quad f_{yy}(0,1) = 6, \quad f_{xy}(0,1) = f_{yx}(0,1) = 1.$$

So, the Taylor expansion is [2]

$$\begin{aligned} f(x,y) &= 1 + x + 3(y-1) + x^2 + 3(y-1)^2 + x(y-1), \\ &= 1 + x^2 + xy + 3y(y-1), \end{aligned}$$

and [2]

$$f(0.1, 1.1) = 1 + (0.1)^2 + (0.1)(1.1) + 3(1.1)(0.1) = 1 + 0.01 + 0.44 = 1.45.$$

3. (a) Using the chain rule we have [4]

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = e^{u+v} f_x + e^{u-v} f_y,$$

and [4]

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = e^{u+v} f_x - e^{u-v} f_y.$$

(b) From (a) we can obtain the 2nd order partial derivatives by using once more the chain rule we have [5]

$$\begin{aligned} \frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}(e^{u+v} f_x + e^{u-v} f_y) = e^{u+v} f_x + e^{u+v}\frac{\partial f_x}{\partial u} + e^{u-v} f_y + e^{u-v}\frac{\partial f_y}{\partial u} \\ &= e^{u+v} f_x + e^{u-v} f_y + e^{u+v}(e^{u+v} f_{xx} + e^{u-v} f_{yx}) + e^{u-v}(e^{u+v} f_{xy} + e^{u-v} f_{yy}) \\ &= e^{u+v} f_x + e^{u-v} f_y + e^{2(u+v)} f_{xx} + e^{2(u-v)} f_{yy} + e^{2u}(f_{xy} + f_{yx}), \end{aligned}$$

and [5]

$$\begin{aligned} \frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v}(e^{u+v} f_x - e^{u-v} f_y) = e^{u+v} f_x + e^{u+v}\frac{\partial f_x}{\partial v} + e^{u-v} f_y - e^{u-v}\frac{\partial f_y}{\partial v} \\ &= e^{u+v} f_x + e^{u-v} f_y + e^{u+v}(e^{u+v} f_{xx} - e^{u-v} f_{yx}) - e^{u-v}(e^{u+v} f_{xy} - e^{u-v} f_{yy}) \\ &= e^{u+v} f_x + e^{u-v} f_y + e^{2(u+v)} f_{xx} + e^{2(u-v)} f_{yy} - e^{2u}(f_{xy} + f_{yx}). \end{aligned}$$

Subtracting the two formulae we trivially see that [2]

$$\frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial v^2} = 2e^{2u}\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial x}\right).$$

4. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition [2]

$$m^2 - 4 = 0 \Rightarrow m = \pm 2.$$

This means that the general solution of the homogeneous equation is of the form [2]

$$y = c_1 e^{2x} + c_2 e^{-2x},$$

therefore we identify

$$u_1(x) = e^{2x}, \qquad u_2(x) = e^{-2x}.$$

For the second part of the problem we will need the Wronskian of these solutions which is [3]

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4.$$

Therefore the Wronskian is indeed nowhere zero.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form [3]

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = \cos(2x), \qquad W(x) = -4,$$

therefore, integrating by parts twice we obtain

$$v_1(x) = \frac{1}{4}\int e^{-2x}\cos(2x)dx = \frac{1}{4}\left[\frac{1}{2}e^{-2x}\sin(2x) - \frac{1}{2}\cos(2x)e^{-2x} - 4v_1(x)\right],$$

which gives

$$v_1(x) = \frac{(-\cos(2x) + \sin(2x))e^{-2x}}{16}.$$

and [4]

$$v_2(x) = -\frac{1}{4}\int e^{2x}\cos(2x)dx = -\frac{1}{4}\left[\frac{1}{2}e^{2x}\sin(2x) + \frac{1}{2}\cos(2x)e^{2x} + 4v_2(x)\right],$$

$$v_2(x) = -\frac{(\cos(2x) + \sin(2x))e^{2x}}{16}.$$

[4]

Hence the general solution of the inhomogeneous equation is [2]

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{\cos(2x)}{8}.$$

with c_1, c_2 being arbitrary constants.