Section A: Calculus

1. (a) Sketch the region of integration in the double integral

$$I = \int_{y=0}^{y=2} dy \int_{x=y/2}^{x=1} \cos(x^2) dx.$$

By changing the order of integration, evaluate I.

(b) The cylindrical coordinates (r, θ, z) are related to the Cartesian coordinates (x, y, z) by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Obtain the Jacobian determinant of the transformation from Cartesian to cylindrical coordinates. Hence use cylindrical coordinates to compute the integral

$$V = \int\int\int_R (x^2 + y^2)^2 dx\, dy\, dz$$

on a region R corresponding to a circular cylinder of radius 1 centered at the origin and located above the $z = 1$ plane and below the $z = 5$ plane.

2. (a) Find and classify the stationary points (maxima, minima and saddle points) of the function

$$f(x, y) = xe^{-x^2+y^2}.$$

(b) Compute the Taylor's expansion of the function

$$f(x, y) = x^2 + xy + y^3.$$

around the point $(0, 1)$ including up to second order terms. Hence estimate the value of the function at the point $(0.1, 1.1)$.

3. Consider the following transformation of coordinates

$$x = e^{u+v}, \qquad y = e^{u-v}$$

in two dimensional space. Let $f(x, y)$ be a twice differentiable function of x and y.

Turn over ...

(a) Use the chain rule to express the first order partial derivatives of f with respect to u and v in terms of the first order partial derivatives of f with respect to x and y.

(b) Likewise, use the chain rule to find $\partial^2 f/\partial u^2$ and $\partial^2 f/\partial v^2$. Hence show that

$$\frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial v^2} = 2e^{2u}\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial x}\right).$$

4. Determine the functions $u_1(x)$, $u_2(x)$ such that $y(x) = c_1 u_1(x) + c_2 u_2(x)$ is the general solution of the following homogeneous second-order differential equation

$$y'' - 4y = 0,$$

where c_1, c_2 are arbitrary constants. Show that the Wronskian of u_1, u_2 is nowhere zero.

Use the method of variation of parameters to find a particular solution of the inhomogeneous second-order differential equation

$$y'' - 4y = \cosh(2x).$$

Hence determine the general solution of this inhomogeneous equation.

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