Section A: Calculus

1. (a) Sketch the region of integration in the double integral

$$I = \int_{y=0}^{y=2} dy \int_{x=y/2}^{x=1} e^{x^2} dx.$$

By changing the order of integration, evaluate I.

(b) The cylindrical coordinates (r, θ, z) are related to the Cartesian coordinates (x, y, z) by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Obtain the Jacobian determinant of the transformation from Cartesian to cylindrical coordinates. Hence use cylindrical coordinates to compute the integral

$$V = \int\int\int_R (x^2 + y^2)^3 dx\, dy\, dz$$

on a region R corresponding to a circular cylinder of radius 1 centered at the origin and located above the $z = 1$ plane and below the $z = 5$ plane.

2. (a) Find and classify the stationary points (maxima, minima and saddle points) of the function

$$f(x, y) = 4xy - x^4 - y^4.$$

(b) Compute the Taylor's expansion of the function

$$f(x, y) = (x + y)e^{x-y}.$$

around the point $(0, 1)$ including up to second order terms. Hence estimate the value of the function at the point $(0.1, 1.1)$.

Turn over...

3. Consider the following transformation of coordinates

$$x = \frac{u^2 - v^2}{2}, \qquad\qquad y = uv$$

in two dimensional space. Let $f(x, y)$ be a twice differentiable function of x and y.

(a) Use the chain rule to express the first order partial derivatives of f with respect to u and v in terms of the first order partial derivatives of f with respect to x and y.

(b) Likewise, use the chain rule to find $\partial^2 f/\partial u^2$ and $\partial^2 f/\partial v^2$. Hence show that

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = (u^2 + v^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

4. Determine the functions $u_1(x)$, $u_2(x)$ such that $y(x) = c_1 u_1(x) + c_2 u_2(x)$ is the general solution of the following homogeneous second-order differential equation

$$y'' - 4y' + 5y = 0,$$

where c_1, c_2 are arbitrary constants. Show that the Wronskian of u_1, u_2 is nowhere zero.

Use the method of variation of parameters to find a particular solution of the inhomogeneous second-order differential equation

$$y'' - 4y' + 5y = \frac{e^{2x}}{\sin x}.$$

Hence determine the general solution of this inhomogeneous equation.

Turn over...

Section B: Linear Algebra

In the following questions, $M(2,2)$ and P_n denote the vector spaces over $\mathbb{R}$ of all real-valued 2×2 matrices and all polynomials of degree at most n with real coefficients respectively.

5. (a) Determine whether the following subsets are subspaces (giving reasons for your answers).

   i. $U = \{(x, y, x+y) \mid x, y \in \mathbb{R}\}$ in $\mathbb{R}^3$

   ii. $V = \{A \in M(2,2) \mid \det A = 1\}$ in $M(2,2)$

   iii. $W = \{p(x) \in P_2 \mid \frac{d}{dx}(p(x)) = 0\}$ in P_2

   (b) Find a basis for the real vector space $\mathbb{R}^3$ containing the vector $(1,2,3)$.

   (c) Do the following sets form a basis for $\mathbb{R}^3$? If not, determine whether they are linearly independent, a spanning set for $\mathbb{R}^3$, or neither.

   i. $\{(5,0,0), (6,2,0), (5,3,1), (3,1,0)\}$

   ii. $\{(1,2,5), (-1,-1,-1)\}$

6. (a) Let V, W be two real vector spaces and let $f : V \to W$ be a linear map. Define what is meant by the image, the kernel, the rank and the nullity of f and state the Rank-Nullity theorem.

   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear map with $\operatorname{rank} f = 3$. Determine whether the map is injective, surjective, both or neither. Justify your answer.

   (c) Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

   $$f(x, y, z) = (2x + y, y + 2z, 0).$$

   i. Show that f is linear.

   ii. Write down the matrix representing f with respect to the standard ordered basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ of $\mathbb{R}^3$.

   iii. Determine whether f is injective, surjective, both or neither and find a basis for the kernel of f and a basis for the image of f.

Turn over...

7. (a) Let A be a real $n \times n$ matrix. Define what is meant by an eigenvector and an eigenvalue for A.

   (b) State the diagonalization theorem for matrices.

   (c) Let $A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$. Use the diagonalization theorem to find an invertible 3×3 matrix P (and P^{-1}) such that $P^{-1}AP$ is diagonal.

   (d) Calculate A^{100}.

8. Consider the real vector space P_2 with real inner product given by

$$\langle p(x), q(x) \rangle = p_0 q_0 + p_1 q_1 + p_2 q_2$$

for $p(x) = p_0 + p_1 x + p_2 x^2, q(x) = q_0 + q_1 x + q_2 x^2 \in P_2$.

   (a) Define the norm of a vector $p(x) = p_0 + p_1 x + p_2 x^2 \in P_2$ with respect to the above inner product. What is the norm of $x^2 + x + 3$?

   (b) When do we say that two vectors $p(x), q(x) \in P_2$ are orthogonal (with respect to the above norm)? Are x^2 and $2x + 5$ orthogonal?

   (c) What is an orthonormal set of vectors in P_2 (with respect to the above norm)? Is $\{x^2, \frac{1}{\sqrt{2}}(1+x), \frac{1}{\sqrt{2}}(1-x)\}$ an orthonormal set? Justify your answer.

   (d) Use the Gram-Schmidt process to construct an orthonormal basis for P_2 starting from the basis

$$\{2 + 3x^2, -1 + 5x^2, 10 - 7x + 2x^2\}$$

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