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CITY UNIVERSITY

London

BSc Honours Degrees in Mathematical Science
BSc Honours Degree in Mathematical Science with Finance and Economics
BSc Honours Degree in Actuarial Science
BSc Honours Degree in Statistical Science with Management Studies

PART II EXAMINATION

Calculus and Linear Algebra

Thursday 25 May 2000 9:00 am – 12:00 pm

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE of the EIGHT questions with not more than THREE questions from either section

If more than FIVE questions are answered, the best FIVE marks will be credited.

Use a separate answer book for each section.

Section A

1. (a) Sketch the region of integration in the double integral

$$I = \int_0^{\pi/2} dy \int_0^y (\cos 2y)\sqrt{1 - k^2 \sin^2 x}\; dx,$$

where k is a constant.
By changing the order of integration, evaluate I.

(b) Find the Jacobian of the transformation $x = v/u$ and $y = v$. Using the variables u, v evaluate the double integral

$$\int \int_S \frac{y^2}{x^2} e^{y/x} dx dy,$$

where S is the region defined by $0 \le x \le 1$, $0 \le y \le x$.

2. (a) Use Taylor's theorem to expand the function

$$f(x, y) = e^{2x+3y}(8x^2 - 6xy + 3y^2)$$

up to second-order terms in the components h, k of the displacements around the origin $(0, 0)$. What can you conclude from the form of the expansion about the nature of the point $(0, 0)$?

(b) Using the method of Lagrange's multipliers, determine the maximum of the function

$$f(x, y, z) = xyz$$

subject to the condition

$$x^3 + y^3 + z^3 = 1,$$

with $x \ge 0, y \ge 0, z \ge 0$.

Turn over ...

3. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y(x) = Ay_1(x) + By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0,$$

where A, B are arbitrary constants. Show that the Wronskian of the functions $y_1(x)$ and $y_2(x)$ is nowhere zero.

Use the method of variation of constants to find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = \frac{2e^{2x}}{\sin\, x}.$$

Hence determine the general solution of this inhomogeneous equation.

4. Assuming that the solution of the differential equation

$$\frac{d^2y}{dx^2} + x^2y = 0$$

can be expressed as a power series

$$\sum_0^\infty a_n x^n,$$

show that the coefficients a_n satisfy the recurrence relation

$$a_{n+2} = \frac{-a_{n-2}}{(n+1)(n+2)}; \quad n = 2, 3, ...$$

Hence determine the first four non-zero terms of a general series solution of the differential equation.

Turn over ...

Section B

5. (a) What are the properties required for a matrix to be in echelon form?

   (b) Two matrices are row equivalent if there is a sequence of elementary row operations which transforms one into the other. Show that every matrix may be transformed into echelon form using only two types of elementary row operation.

   (c) Let M' be the (not necessarily unique) transform of a matrix M into echelon form using the two types of operation in (b). If M is square, show that its determinant is given, up to sign, by the product of the diagonal entries in any such M'.

6. (a) Recall that an *orthogonal matrix* is a matrix P such that $P^t P = 1$. Showing all your calculations explicitly, determine an orthogonal matrix P such that $P^{-1}AP$ is diagonal, where

$$A = \begin{pmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{pmatrix}.$$

   (b) Make three separate observations on the matrix A above, each of which is on its own enough to demonstrate that the matrix is singular.

Turn over...

7. Let

$$M = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

Determine the eigenvalues of this matrix, and determine a basis for each of the corresponding eigenspaces.

8. Let P_j denote the price of a certain commodity on the j^{th} day of commodity trading in some market. You are given the prices $P_1, P_2, \ldots, P_n$, i.e. the prices on days 1 through n. Prove in case $n = 5$ that there exists a polynomial $f(x)$ of degree at most $n-1$ such that $f(j) = P_j$ for each given P_j. Note that it is NOT necessary to determine $f(x)$.

   Having performed such an analysis on data for each of two separate commodities an analyst finds that the polynomial for commodity A, let us call it f_A, is of degree $n-1$, while there is a polynomial f_B of degree $m < n-1$ fitting the data for commodity B. Which of these polynomials would you expect to be likely to give a better prediction of the corresponding commodity's price on day $j+1$? Give reasons for your answer.

Internal Examiners:	Professor J. Mathon
	Professor P.P. Martin
External Examiners:	Professor J.H. Merkin
	Professor S.A. Robertson