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CITY UNIVERSITY

London

BSc Honours Degrees in Mathematical Science
BSc Honours Degree in Mathematical Science with Finance and Economics
BSc Honours Degree in Mathematics and Finance

PART II EXAMINATION

Calculus and Linear Algebra

9 June 2003 9:00 am – 12:00 pm

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE of the EIGHT questions with not more than THREE questions from either section

If more than FIVE questions are answered, the best FIVE marks will be credited.

Use a separate answer book for each section.

Section A: Calculus

1. (a) Sketch the region of integration in the double integral

$$I = \int_0^1 dy \int_y^1 \frac{e^x}{x}\, dx.$$

By changing the order of integration, evaluate I.

(b) The region R in the positive octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) is bounded by the surface $y = 4x^2$ and by the planes $x = 0$, $y = 4$, $z = 0$ and $z = 2$. Evaluate the volume integral

$$\iiint_R 2x\, dx\, dy\, dz.$$

2. (a) Find and classify the stationary points of the function

$$f(x, y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x.$$

(b) Use Taylor's theorem to expand the function $f(x, y) = (x + y)e^{(x-y)}$ up to second-order terms in the components h, k of the displacements around the point $(-1, -1)$. Hence estimate the value of the function f at the point $(-0.9, -1.05)$.

Turn over ...

3. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y(x) = Ay_1(x) + By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - y = 0,$$

where A, B are arbitrary constants. Show that the Wronskian of the functions $y_1(x)$ and $y_2(x)$ is nowhere zero.

Use the method of variation of constants to find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - y = \frac{1}{e^x + 1}.$$

Hence determine the general solution of this inhomogeneous equation.

4. (a) Use the transformation $x = r\cos\theta$, $y = r\sin\theta$ to express partial derivatives with respect to x, y in terms of partial derivatives with respect to r, θ.

If V is a differentiable function of x, y, show that

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = \left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \theta}\right)^2.$$

(b) Given that $F(x, y, z) = 0$ defines z implicitly as a function of x and y, derive the formulae for $\partial z/\partial x$ and $\partial z/\partial y$ in terms of partial derivatives of F.

If $z\tan x - xy^2z^3 = 2xyz$, determine $\partial z/\partial x$ and $\partial z/\partial y$.

Turn over ...

Section B: Linear Algebra

5. (a) Let A be the matrix

$$\begin{pmatrix} 1 & -2 & 2 \\ 8 & 11 & -8 \\ 4 & 4 & -1 \end{pmatrix}.$$

By finding a basis of eigenvectors, determine an invertible matrix P and its inverse P^{-1} such that $P^{-1}AP$ is diagonal.

(b) State the Cayley-Hamilton theorem, and verify it for the above matrix.

6. (a) Determine which of the following sets are subspaces of $\mathbb{R}^n$ (giving reasons for your answers).

(i) $A = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 = 0 \text{ or } x_n = 0\}$,

(ii) $B = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 + 2x_n = 0\}$,

(iii) $C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1\}$.

(b) For each of the following sets, either prove or disprove that it is a basis for $\mathbb{R}^3$ (you should state clearly any theorems or other standard results that you use). For those sets which are not bases, determine whether they are linearly independent, a spanning set, or neither.

(i) $S_1 = \{(1, 2, 3), (2, 3, 4)\}$,

(ii) $S_2 = \{(5, 3, 2), (8, 1, 4), (2, 3, 6)\}$,

(iii) $S_3 = \{(5, 8, 11), (1, 1, 1), (1, -1, -3)\}$.

(c) Is the space spanned by the set S_1 a subspace of the space spanned by S_3? Give reasons for your answer.

Turn over ...

7. (a) State carefully the definition of a real inner product, and say what it means for two vectors to be orthogonal.

   (b) Let $M(2,2)$ be the (real) vector space of all real 2×2 matrices, and let

$$E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \; E_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Form an orthonormal basis of $M(2,2)$ from these elements with respect to the inner product given by $\langle A, B \rangle = \operatorname{tr}(B^T A)$. (You may assume that $\{E_1, E_2, E_3, E_4\}$ is a basis of $M(2,2)$.)

8. (a) Let $P_n(x)$ denote the real vector space of all polynomials of degree at most n in x with real coefficients. Determine which of the following maps are linear (giving reasons for your answers).

   (i) $f : P_2(x) \longrightarrow P_2(x) \qquad p(x) \longmapsto p(x+1)$,

   (ii) $f : P_3(x) \longrightarrow P_2(x) \qquad p(x) \longmapsto p(1)p(2) + \frac{d}{dx}(p)(x)$,

   (iii) $f : P_2(x) \longrightarrow P_2(x) \qquad p(x) \longmapsto xp(0) + p(1)$.

   (b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$, and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear map given on the standard basis by

$$f(\mathbf{e}_1) = 2\mathbf{e}_2 + \mathbf{e}_3 \quad \text{and} \quad f(\mathbf{e}_2) = \mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3.$$

   Determine the matrix of this map with respect to the bases $\{3\mathbf{e}_1 - \mathbf{e}_2, 2\mathbf{e}_1 + 3\mathbf{e}_2\}$ of $\mathbb{R}^2$ and $\{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3\}$ of $\mathbb{R}^3$.

   (c) Is the map f in (b) an isomorphism? Give a reason for your answer.

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