Section A: Calculus

1. (a) Sketch the region of integration in the double integral

$$I = \int_0^2 dy \int_{y/2}^1 e^{x^2}\, dx.$$

By changing the order of integration, evaluate I.

(b) Show that the equation of the semi-circle $x^2 + y^2 - ay = 0, \quad x \geq 0$ in the polar coordinates takes the form

$$r = a \sin\theta; \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Hence using the cylindrical coordinates find the volume of the solid that is inside of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

above the xy-plane, and inside of the vertical cylinder $x^2 + y^2 - ay = 0, \quad x \geq 0$.

2. (a) Find and classify the stationary points of the function

$$f(x, y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x.$$

(b) Use Taylor's theorem to expand the function $f(x, y) = (x + y)e^{(x-y)}$ up to second-order terms in the components h, k of the displacements around the point $(-1, -1)$. Hence estimate the value of the function f at the point $(-0.9, -1.05)$.

Turn over ...

3. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y(x) = Ay_1(x) + By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + 4y = 0,$$

where A, B are arbitrary constants. Show that the Wronskian of the functions $y_1(x)$ and $y_2(x)$ is nowhere zero.

Use the method of variation of constants to find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + 4y = \tan(2x).$$

Hence determine the general solution of this inhomogeneous equation.

4. (a) Use the transformation $x = r\cos\theta, y = r\sin\theta$ to express partial derivatives with respect to x, y in terms of partial derivatives with respect to r, θ.

   (b) If V is a differentiable function of x, y, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}.$$

Turn over ...

Section B: Linear Algebra

In the following questions $M(2,2)$ and P_n denote the vector spaces over $\mathbb{R}$ of all real-valued 2×2 matrices and of all polynomials of degree at most n with real coefficients respectively.

5. (a) Determine which of the following sets are subspaces (giving reasons for your answers).

   (i) $A = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq x_2 \geq 0\}$.

   (ii) $B = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,2) : a + 2b - c - d = 0\}$.

   (iii) $C = \{f(x) \in P_n : f(0) + f(1) = 1\}$.

   (b) For each of the following sets, either prove or disprove that it is a basis for $\mathbb{R}^3$ (you should state clearly any theorems or other standard results that you use). For those sets which are not bases, determine whether they are linearly independent, a spanning set for $\mathbb{R}^3$, or neither.

   (i) $S_1 = \{(7,9,3), (0,0,1)\}$.

   (ii) $S_2 = \{(2,1,0,-1), (1,1,1,1), (2,3,6,8)\}$.

   (iii) $S_3 = \{(3,5,1), (2,1,1), (1,1,8)\}$.

   (iv) $S_4 = \{(0,0,0), (1,0,1), (0,-1,0)\}$.

   (c) Is it always the case that if S and T are bases for a vector space V then so is their intersection $S \cap T$?

6. Consider the following elements of $M(2,2)$:

$$F_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}, F_3 = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}.$$

   (a) Show that $\langle A, B \rangle = \mathrm{tr}(B^T A)$ defines an inner product on $M(2,2)$.

   (b) Verify that $\{F_1, F_2, F_3\}$ is an orthogonal set with respect to the above inner product. Write down an orthonormal set which can be obtained from this one.

   (c) Let G be the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

   Using the fact that $\{F_1, F_2, F_3, G\}$ is a basis for $M(2,2)$ (or otherwise), find a matrix F_4 such that $\{F_1, F_2, F_3, F_4\}$ is an orthonormal basis for $M(2,2)$.

Turn over ...

7. Let A be the matrix

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

State the diagonalisation theorem, and use it to determine matrices P, P^{-1} and D such that $D = P^{-1}AP$ is diagonal. How many different diagonal matrices D can be obtained in this manner?

8. (a) Determine which of the following maps are linear (giving reasons for your answers).

   (i) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad (x_1, x_2, x_3) \longmapsto (x_1 + x_2, x_2x_3, x_2).$

   (ii) $f : P_2 \longrightarrow P_3 \qquad p(x) \longmapsto p(2x + 1).$

   (iii) $f : M(2, 2) \longrightarrow \mathbb{R} \qquad A \longmapsto \mathrm{tr}(A).$

   (b) Define the rank and nullity of a linear map, and state carefully a theorem that relates the two.

   (c) Let $f : P_n \longrightarrow P_n$ be the map

$$f : p(x) \longmapsto p(x) - p(1).$$

   Find bases for the image and kernel of f, stating carefully any theorems that you use. (You may assume that f is linear, and may use without proof that $\dim P_n = n + 1$.)

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