Section A: Calculus

1. (a) Sketch the region of integration in the double integral

$$I = \int_0^1 dy \int_y^1 cos(\frac{1}{2}\pi x^2)dx.$$

By changing the order of integration, evaluate I.

(b) Find the Jacobian of the transformation of coordinates

$$x = rcos\theta,\ y = rsin\theta,\ z = z,$$

where

$$0 \leq \theta \leq 2\pi,\ 0 \leq r < \infty,\ -\infty < z < \infty.$$

Using the coordinates $r,\ \theta,\ z$, determine the mass of the solid bounded by the cone $z^2 = x^2 + y^2$, $z \geq 0$ and the cylinder $x^2 + y^2 = a^2$, given that the density of the solid is defined by the function $(x^2 + y^2)z$.

2. (a) Given that $x = cos(t), y = 2sin(t)$ and $f(x, y) = e^{-(x^2+y^2)}$, use partial differentiation to find df/dt in terms of t.

(b) A change of variables $(u, v) \longmapsto (x, y)$ is defined by

$$x = \frac{1}{2}(u + v),\ y = \frac{1}{4}(u^2 + v^2).$$

If $f(x, y)$ is a twice differentiable function and $f(x(u, v), y(u, v)) = F(u, v)$, show that

$$\frac{\partial F}{\partial u} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$$

$$\frac{\partial F}{\partial v} = -v\frac{\partial f}{\partial x} + u\frac{\partial f}{\partial y}.$$

Hence find expressions for $\partial^2 F/\partial u^2$ and $\partial^2 F/\partial v^2$.

Show that if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

then

$$\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = 0.$$

Turn over ...

3. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y(x) = Ay_1(x) + By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0,$$

where A, B are arbitrary constants. Show that the Wronskian of the functions $y_1(x)$ and $y_2(x)$ is nowhere zero.

Use the method of variation of constants to find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = \frac{e^{2x}}{\sin x}.$$

Hence determine the general solution of this inhomogeneous equation.

4. (a) (a) Use Taylor's theorem to expand the function

$$f(x, y) = e^{2x+3y}(8x^2 - 6xy + 3y^2)$$

up to second-order terms in the components h, k of the displacements around the origin $(0, 0)$. What can you conclude from the form of the expansion about the nature of the point $(0, 0)$?

(b) (b) Using the method of Lagrange's multipliers, determine the maximum of the function

$$f(x, y, z) = xyz$$

subject to the condition

$$x^3 + y^3 + z^3 = 1,$$

with $x \geq 0, y \geq 0, z \geq 0$.

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Section B: Linear Algebra

In the following questions $M(m, n)$ and P_n denote respectively the vector spaces over $\mathbb{R}$ of all real-valued $m \times n$ matrices and of all polynomials of degree at most n with real coefficients.

5. (a) If p and q are polynomials in P_2, write $p(x) = p_0 + p_1x + p_2x^2$ and $q(x) = q_0 + q_1x + q_2x^2$. Determine which (if any) of the following are inner products on P_2, giving reasons for your answers.

   (i) $\langle p, q\rangle = p_0q_0 + p_2q_2$.

   (ii) $\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$.

   (iii) $\langle p, q\rangle = p_1q_0 + p_2q_1 + p_0q_2$.

   (iv) $\langle p, q\rangle = \int_0^1 (x - 1)p(x)q(x)dx$.

   (b) Verify that the elements

   $$\{(1, 1, 0, 0), (1, -1, 2, 0), (1, -1, -1, 3), (1, -1, -1, -1)\}$$

   from $\mathbb{R}^4$ form an orthogonal set with respect to the usual inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{4} x_iy_i$, and construct an orthonormal set from them.

   (c) Explain carefully (without any further calculations) why the orthonormal set in the preceding part must be a basis for $\mathbb{R}^4$.

6. (a) Determine which (if any) of the following maps are linear, giving reasons for your answers.

   (i) $f : P_2 \longrightarrow P_3$ given by $p(x) \longmapsto \int p(x)dx + x$.

   (ii) $f : M(2, 2) \longrightarrow M(3, 3)$ given by $M \longmapsto (\det M)I$

   where I is the identity matrix in $M(3, 3)$.

   (b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$, and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear map given on the standard basis by

   $$f(\mathbf{e}_1) = a\mathbf{e}_1 + b\mathbf{e}_2 \quad \text{and} \quad f(\mathbf{e}_2) = c\mathbf{e}_1 + d\mathbf{e}_2 + e\mathbf{e}_3$$

   for some fixed numbers a, b, c, d, and e. Determine the matrix of this map with respect to the bases $\{\mathbf{e}_1 + 2\mathbf{e}_2, 3\mathbf{e}_1 - \mathbf{e}_2\}$ of $\mathbb{R}^2$ and $\{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1\}$ of $\mathbb{R}^3$.

   (c) What are the possible dimensions of $\mathrm{Im}(f)$ in part (b)? Give examples of maps (i.e. choose values of $a, b, \dots, e$) having each possible dimension.

Turn over ...

7. (a) Let U and V be subspaces of a vector space W. Which of the following are also always subspaces of W? Give reasons for your answers.

   (i) $U \cap V = \{w \in W : w \in U \text{ and } w \in V\}$.

   (ii) $U \cup V = \{w \in W : w \in U \text{ or } w \in V\}$.

   (iii) $U + V = \{w \in W : w = u + v \text{ for some } u \in U \text{ and } v \in V\}$.

   (b) State carefully the definitions of linearly independent and of spanning sets.

   (c) For each of the following sets, either prove or disprove that it is a basis for P_2 (you should state clearly any theorems or other standard results that you use). For those sets which are not bases, determine whether they are linearly independent, a spanning set, or neither.

   (i) $S_1 = \{x, 2x\}$.

   (ii) $S_2 = \{x, x^2 + 1, x^2 + 2x + 3\}$.

   (iii) $S_3 = \{x^2 + x + 3, 2x^2 - x - 1, x^2 - 5x - 11\}$.

8. (a) State carefully the definitions of symmetric and orthogonal matrices.

   (b) Is it possible for a matrix to be both symmetric and orthogonal? Give an example of such a matrix, or explain why it is impossible.

   (c) Let A be the matrix

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

   By constructing a *suitable* basis of eigenvectors, find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$.

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