Section A: Calculus

1. (a) Sketch the region of integration in the double integral

$$I = \int_{y=0}^{y=3} dy \int_{x=y/3}^{x=1} x e^{x^3} \, dx.$$

By changing the order of integration, evaluate I.

(b) The cylindrical coordinates (r, θ, z) are related to the Cartesian coordinates (x, y, z) by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Obtain the Jacobian determinant of the transformation from Cartesian to cylindrical coordinates. Hence use cylindrical coordinates to obtain the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$, the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3$ for $y > 0$.

2. (a) Find and classify the stationary points (maxima, minima and saddle points) of the function

$$f(x, y) = ax^2 + bxy + cy^2,$$

if a, b, c are all positive, real constants with $b^2 < 4ac$. How would the nature of the stationary points change if a was instead negative with b, c positive as before?

(b) Compute the Taylor's expansion of the function

$$f(x, y) = xye^{x+y}.$$

around the point $(0, 0)$ including up to second order terms. What can you conclude from the form of the expansion about the nature of the point (0,0)?

Turn over...

3. (a) Let $z = f(x, y)$ be a real function of the two independent variables x and y which is defined implicitly by means of a constraint of the form $G(x, y, z) = 0$. Obtain formulae for the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ in terms of the partial derivatives G_x, G_y, G_z. Give all the details of the derivation. Use these formulae to compute $\partial z/\partial x$ and $\partial z/\partial y$ if

$$G(x, y, z) = \sin(xyz).$$

(b) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = 6x + y$ subject to the constraint $x^2 - y^2 = 1$. Compute also the corresponding values of the Lagrange multiplier.

4. Determine the functions $u_1(x)$, $u_2(x)$ such that $y(x) = c_1u_1(x) + c_2u_2(x)$ is the general solution of the following homogeneous second-order differential equation

$$y'' + 8y' + 25y = 0,$$

where c_1, c_2 are arbitrary constants. Show that the Wronskian of u_1, u_2 is nowhere zero.

Use the method of variation of parameters to find a particular solution of the inhomogeneous second-order differential equation

$$y'' + 8y' + 25y = e^{-2x}.$$

Hence determine the general solution of this inhomogeneous equation.

Turn over...

Section B: Linear Algebra

In the following questions, $M(2,2)$ and P_n denote the vector spaces over $\mathbb{R}$ of all real-valued 2×2 matrices and all polynomials of degree at most n with real coefficients respectively.

5. (a) Determine whether the following subsets are subspaces (giving reasons for your answers).

   i. $U = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,2) \mid a + d = b + c\}$ in $M(2,2)$

   ii. $V = \{p(x) \in P_3 \mid p(-2) = 0\}$ in P_3

   iii. $W = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \ldots \geq x_n\}$ in $\mathbb{R}^n$

   (b) Find a basis for the real vector space $M(2,2)$. What is the dimension of $M(2,2)$?

   (c) Do the following sets form a basis for $\mathbb{R}^3$? If not, determine whether they are linearly independent, a spanning set for $\mathbb{R}^3$, or neither.

   i. $\{(1,0,2), (1,2,0), (0,1,2), (2,2,2)\}$.

   ii. $\{(5,0,0), (2,1,-3), (-1,4,0)\}$.

6. (a) Let V, W be real vector spaces. Define what it means for a map $f : V \to W$ to be linear.

   (b) Is there a linear map $f : \mathbb{R}^2 \to M(2,2)$ satisfying $f(1,0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $f(1,1) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $f(2,1) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$? Justify your answer.

   (c) Suppose $f : \mathbb{R}^2 \to M(2,2)$ is a linear map satisfying $f(1,0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $f(1,1) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Find $f(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

   (d) Define what is meant by the image, the kernel, the rank and the nullity of a linear map and state carefully the Rank-Nullity theorem.

   (e) Consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^4$ given by

   $$f(x,y,z) = (x + y, y + z, x + y, y + z)$$

   for all $(x,y,z) \in \mathbb{R}^3$. Determine whether f is injective, surjective, both or neither and find a basis for the kernel of f and a basis for the image of f.

7. (a) Define what is meant by an eigenvector and an eigenvalue for a real $n \times n$ matrix.

   (b) State carefully the diagonalization theorem for matrices.

   (c) Show that the matrix $A = \begin{pmatrix} -4 & 0 & -2 \\ -4 & -2 & -4 \\ 4 & 0 & 2 \end{pmatrix}$ is diagonalizable and hence find an invertible 3×3 matrix P (and P^{-1}) such that $P^{-1}AP$ is diagonal.

8. (a) Show that

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx$$

   for all $p(x), q(x) \in P_2$ defines a real inner product on the vector space P_2.

   (b) Define the norm of a polynomial $p(x) \in P_2$ with respect to the above inner product. What is the norm of $p(x) = x$?

   (c) When do we say that two polynomials $p(x), q(x) \in P_2$ are orthogonal (with respect to the above inner product)? Are $x + 1$ and x^2 orthogonal?

   (d) What is an orthonormal set of polynomials in P_2 (with respect to the above inner product)? Check that

$$\{p_1(x) = \frac{1}{\sqrt{2}}, p_2(x) = \frac{\sqrt{3}}{\sqrt{2}}x\}$$

   is an orthonormal set.

   (e) Using the fact that $\{p_1(x), p_2(x), x^2\}$ is a basis for P_2, find a polynomial $p_3(x)$ such that $\{p_1(x), p_2(x), p_3(x)\}$ is an orthonormal basis for P_2.

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