No: 615.53

CITY UNIVERSITY

LONDON

BSc Honours Degree in Actuarial Science
BSc Honours Degrees in Mathematical Science
BSc Honours Degree in Statistical Science with Management Studies

PART II EXAMINATION

CALCULUS & LINEAR ALGEBRA

Monday 15 June 1998 1.00 pm - 4.00 pm

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE of the EIGHT questions.

If more than FIVE questions are answered, the best FIVE marks will be credited.

Use a separate answer book for each section.

Section A

1. (a) Sketch the region of integration in the double integral

$$I = \int_0^1 dy \int_y^1 cos(\frac{1}{2}\pi x^2)dx.$$

By changing the order of integration, evaluate I.

(b) Find the Jacobian of the transformation of coordinates

$$x = rcos\theta, \ y = rsin\theta, \ z = z,$$

where

$$0 \leq \theta \leq 2\pi, \ 0 \leq r < \infty, \ -\infty < z < \infty.$$

Using the coordinates $r, \ \theta, \ z$, determine the mass of the solid bounded by the cone $z^2 = x^2 + y^2$, $z \geq 0$ and the cylinder $x^2 + y^2 = a^2$, given that the density of the solid is defined by the function $(x^2 + y^2)z$.

2. (a) Use Taylor's theorem to expand the function $f(x, y) = (x^2 + y^2 - xy)e^{(x+y)}$ up to second-order terms in the components h, k of the displacements around the point $(1, -1)$. Hence estimate the value of the function f at the point $(1.1, -0.95)$.

(b) Using the method of Lagrange multipliers, find the shortest distance from the origin $(0, 0)$ to the curve

$$x^2 + 4xy + y^2 - 4 = 0.$$

3. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y = Ay_1(x) + By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - y = 0,$$

where A, B are arbitrary constants. Show that the Wronskian of the functions $y_1(x)$ and $y_2(x)$ is nowhere zero.

Turn over ...

Use the method of variation of constants to find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - y = \frac{2}{e^x + e^{-x}}.$$

Hence determine the general solution of this inhomogeneous equation.

4. (a) Given that $x = cos(t), y = 2sin(t)$ and $f(x, y) = e^{-(x^2+y^2)}$, use partial differentiation to find df/dt in terms of t.

(b) A change of variables $(u, v) \longmapsto (x, y)$ is defined by

$$x = \frac{1}{2}(u^2 - v^2),\ y = uv.$$

If $f(x, y)$ is a twice differentiable function and $f(x(u, v), y(u, v)) = F(u, v)$, show that

$$\frac{\partial F}{\partial u} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$$

$$\frac{\partial F}{\partial v} = -v\frac{\partial f}{\partial x} + u\frac{\partial f}{\partial y}.$$

Hence find expressions for $\partial^2 F/\partial u^2$ and $\partial^2 F/\partial v^2$.
Show that if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

then

$$\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = 0.$$

Turn over ...

Section B

5. Two maps s_0 and s_1 on $\mathbb{R}^2$ are defined by

$$s_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$s_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -c \end{pmatrix}$$

where $c \in \mathbb{N}$.

(a) Show that $s_0 = s_1 = 1_{\mathbb{R}^2}$.

(b) A subset T of $\mathbb{R}^2$ is said to be <u>fixed</u> under these maps if $s_0 t \in T$ and $s_1 t \in T$ for all $t \in T$. Show that for each $v \in \mathbb{R}^2$ there is an unique subset $\mathcal{O}_v$ of $\mathbb{R}^2$ fixed under s_0 and s_1 such that

(i) $v \in \mathcal{O}_v$;

(ii) $\mathcal{O}_v \subseteq T$ for every subset T of $\mathbb{R}^2$ which contains v and is fixed under s_0 and s_1.

(c) Sketch a graph of the part of $\mathcal{O}_v$ in the region

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid -15 \leq x \leq 15,\ -15 \leq y \leq 15 \right\}$$

in case $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $c = 1$.

6. (a) State the Cayley-Hamilton theorem.

(b) For

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

determine an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

(c) Determine the eigenvalues and four corresponding linearly independent eigenvectors of

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Turn over...

7. Give a matrix method for obtaining a sequence (r_n) of rational numbers with $r_n \to 1 + \sqrt{2}$ as $n \to \infty$. (You should explain the procedure in detail, but it is not necessary to prove convergence explicitly.)

8. Let V be the vector space (over the field $\mathbb{R}$) of all real 2×2 matrices and let

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(a) Verify that the set $\{E_1, E_2, E_3, E_4\}$ is a basis for V.

(b) Show that $< u, v > = \text{trace}(v^t u)$ defines an inner product on V.

(c) Find $M \in V$ such that $\{E_1, E_2, E_3, M\}$ is an orthonormal basis with respect to $<, >$.

Internal Examiner: Prof.P.P.Martin