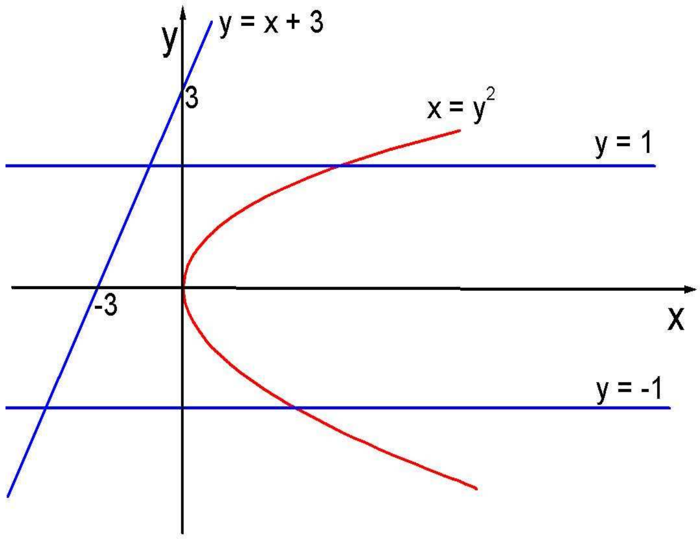
Calculus 2008: Solutions

1. (a) The integration region is shown in the figure below:



$\boxed{\text{3 points}}$

The integral is simply

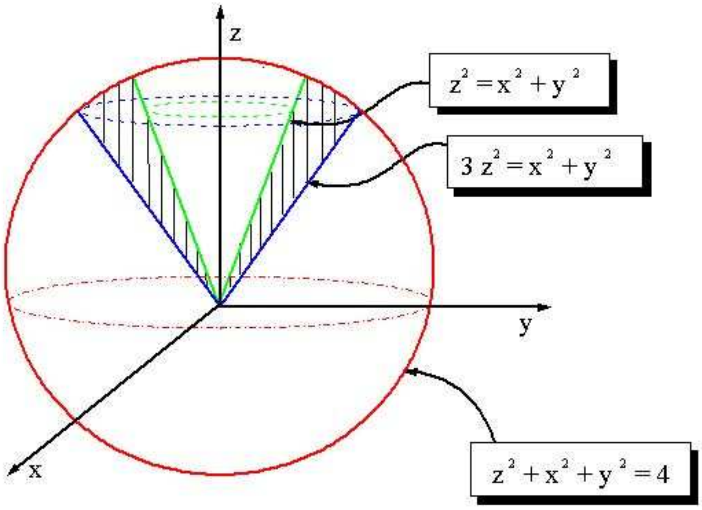
$$\int_{y=-1}^{y=1}\int_{x=y-3}^{x=y^2}(x+y)dxdy = \int_{y=-1}^{y=1}\left[\frac{x^2}{2}+yx\right]_{x=y-3}^{x=y^2} dy$$

$$= \int_{y=-1}^{y=1}\left(\frac{y^4}{2}+y^3-\frac{(y-3)^2}{2}-y(y-3)\right)dy = \left[\frac{y^5}{10}+\frac{y^4}{4}-\frac{(y-3)^3}{6}-\frac{y^3}{3}+\frac{3y^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{10}+\frac{1}{4}-\frac{(-2)^3}{6}-\frac{1}{3}+\frac{3}{2}+\frac{1}{10}-\frac{1}{4}+\frac{(-4)^3}{6}-\frac{1}{3}-\frac{3}{2} = \frac{1}{5}-\frac{2}{3}+\frac{4}{3}-\frac{32}{3} = \frac{1}{5}-10 = -\frac{49}{5}.$$

$\boxed{\text{7 points}}$

(b) We start by getting an idea of how the integration region looks like: We have a sphere of radius 2 and two different cones. The three surfaces are centered at the origin and the only subtlety is perhaps to figure out which cone is inside which. In order to do that you can look at "sections" of each cone and compare them. For example, the intersection of the first cone with the plane $z = 1$ is the circle $x^2 + y^2 = 1$. The intersection of the second cone with the same plane is the circle $x^2+y^2 = 3$, which is clearly bigger. This will be true for any choice of z. Therefore, we can conclude that the cone $z = \sqrt{x^2 + y^2}$ is inside the cone $z = \sqrt{(x^2 + y^2)/3}$. The three surfaces will look more or less like in the picture in the next page, where the integration region is the volume in between the two cones.



The integral which we have to compute is:

$$V = \int\int\int_R dxdydz = \int\int\int_R r^2 \sin\phi \, dr d\theta d\phi.$$

Now we need to find R is spherical coordinates

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi.$$

In these coordinates the equations of the sphere and the cones become

$$x^2 + y^2 + z^2 = 4 \quad \Leftrightarrow \quad r = 2,$$

$$z = \sqrt{x^2 + y^2} \quad \Leftrightarrow \quad r\cos\phi = r\sin\phi \quad \Leftrightarrow \quad \phi = \frac{\pi}{4},$$

and

$$z = \sqrt{(x^2 + y^2)/3} \quad \Leftrightarrow \quad \sqrt{3}r\cos\phi = r\sin\phi \quad \Leftrightarrow \quad \phi = \frac{\pi}{3}.$$

Therefore

$$R = \{(r, \theta, \phi) \,|\, 0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \quad \frac{\pi}{4} \le \phi \le \frac{\pi}{3}\}.$$

Then the volume is $\boxed{\text{5 points}}$

$$V = \int_{r=0}^{r=2}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{3}} r^2 \sin\phi \, d\phi dr d\theta = \int_{r=0}^{r=2} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{3}} \sin\phi \, d\phi.$$

In this case, since all integration limits are constants, the order of integration does not matter and we can do each integral separately and multiply the values at the end:

$$\int_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{3}} \sin\phi \, d\phi = [-\cos\phi]_{\pi/4}^{\pi/3} = \frac{1}{\sqrt{2}} - \frac{1}{2} = \frac{\sqrt{2}-1}{2},$$

$$\int_{\theta=0}^{\theta=2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi,$$

$$\int_{r=0}^{r=2} r^2 dr = \left[\frac{r^3}{3}\right]_0^2 = \frac{8}{3}.$$

The volume is

$$V = \frac{\sqrt{2}-1}{2} \times 2\pi \times \frac{8}{3} = \frac{8\pi(\sqrt{2}-1)}{3}.$$

5 points

2. (a) First of all we need to find the points at which the first order partial derivatives vanish. These derivatives are

$$f_x = y\cos x, \qquad f_y = \sin x.$$

Then

1 point

$$f_x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad y = 0,$$

and

$$f_y = 0 \quad \Leftrightarrow \quad x = 0, \pi, 2\pi.$$

The two derivatives vanish simultaneously at the points $(0,0)$, $(\pi,0)$ and $(2\pi,0)$. 3 points

The second order partial derivatives are:

$$f_{xx} = -y\sin x, \qquad f_{yy} = 0,$$

$$f_{xy} = \cos x.$$

For the point $(0,0)$ we find that: 1 point

$$f_{xx}f_{yy} - f_{xy}^2 = -1 < 0,$$

and we conclude that the point is a saddle point. Similarly for the points $(\pi,0)$ and $(2\pi,0)$ we find:

$$f_{xx}f_{yy} - f_{xy}^2 = -1 < 0,$$

so that they are also saddle points. 5 points

(b) The differential is by definition:

$$dg = g_x dx + g_y dy,$$

with

$$g_x = \frac{-2x}{(x^2+y^2-1)^2}, \qquad g_y = \frac{-2y}{(x^2+y^2-1)^2},$$

At the point $(1,-1)$ the value of the function is simply $g(1,-1) = 1$ and the derivatives are $g_x(1,-1) = -2$ and $g_y(1,-1) = 2$. Therefore the estimated value of $g(1.01,-1.003)$ is given by 4 points

$$g(1.01,-1.003) \approx g(1,-1) + g_x(1,-1)dx + g_y(1,-1)dy = 1 - 2(1.01-1) + 2(-1.003+1) = 0.974$$

The exact value of the function at this point is $g(1.01,-1.003) = 0.97165$ which is very close to the value above. 6 points

3. (a) Let the point (x, y) be a maximum of $f(x, y)$, then

$$df = f_x dx + f_y dy = 0,$$

since the first order partial derivatives always vanish at a maximum point. $\boxed{\text{1.5 points}}$

Since $\phi(x, y) = 0$, then it follows trivially that

$$d\phi = \phi_x dx + \phi_y dy = 0,$$

and therefore we can also write that $\boxed{\text{1.5 points}}$

$$d(f + \lambda\phi) = df + \lambda d\phi = 0,$$

where λ is an arbitrary constant which we call **Lagrange's multiplier**. $\boxed{\text{1 points}}$

The previous equation is equivalent to

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right) dy = 0.$$

$\boxed{\text{2 points}}$

Since λ is arbitrary we can choose it so that

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0.$$

which implies

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0.$$

Therefore, we have now a set of 3 equations, for 3 unknowns namely

$$\phi(x, y) = f_x + \lambda\phi_x = f_y + \lambda\phi_y = 0.$$

$\boxed{\text{2 points}}$

(b) In this case the equations above become

$$x + y - 9 = 0, \qquad 3x^2 + \lambda = 0, \qquad 3y^2 + \lambda = 0,$$

Therefore we obtain, $\boxed{\text{2 points}}$

$$3x^2 = 3y^2 = -\lambda,$$

which implies $x = \pm y$ and $x = \pm\sqrt{-\lambda/3}$.

This gives us 4 different kinds of solutions: $(x, y) = (\pm\sqrt{-\lambda/3}, \pm\sqrt{-\lambda/3}), (\pm\sqrt{-\lambda/3}, \mp\sqrt{-\lambda/3})$.

Plugging these solutions into the 1st equation we find that the two last solutions are not consistent with it. Therefore, only the solutions $(x, y) = (\pm\sqrt{-\lambda/3}, \pm\sqrt{-\lambda/3})$ are sensible. $\boxed{\text{4 points}}$

The solution $x = y = \sqrt{\frac{-\lambda}{3}}$ gives

$$2\sqrt{\frac{-\lambda}{3}} = 9 \qquad \Leftrightarrow \qquad \lambda = -\frac{3^5}{4},$$

which implies $(x, y) = (9/2, 9/2)$. When putting the solution $x = y = -\sqrt{\frac{-\lambda}{3}}$ into the equation $x + y = 9$ we obtain the same values of x, y, λ as in the previous case. $\boxed{\text{3 points}}$

Therefore, the point $(9/2, 9/2)$ is the maximum of $f(x, y)$. $\boxed{\text{3 points}}$

4. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 - 1 = 0 \Rightarrow m = \pm 1.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 e^x + c_2 e^{-x},$$

therefore we identify

$$u_1(x) = e^x, \qquad u_2(x) = e^{-x}.$$

For the second part of the problem we will need the Wronskian of these solutions which is $\boxed{\text{4 points}}$

$$\begin{aligned} W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \\ &= -1 - 1 = -2. \end{aligned}$$

$\boxed{\text{3 points}}$

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = \frac{1}{1+e^{-x}}, \qquad W(x) = -2,$$

therefore $\boxed{\text{3 points}}$

$$v_1(x) = \frac{1}{2}\int \frac{e^{-x}}{1+e^{-x}}dx = -\frac{1}{2}\ln(1+e^{-x}).$$

and similarly $\boxed{\text{4 points}}$

$$\begin{aligned} v_2(x) &= -\frac{1}{2}\int \frac{e^x}{1+e^{-x}}dx = -\frac{1}{2}\int \frac{1}{e^{-x}(1+e^{-x})} = -\frac{1}{2}\int \left(\frac{1}{e^{-x}} - \frac{1}{1+e^{-x}}\right) \\ &= -\frac{e^x}{2} + \frac{1}{2}\int \frac{1}{1+e^{-x}}dx = -\frac{e^x}{2} + \frac{1}{2}\int \frac{e^x}{e^x+1}dx = -\frac{e^x}{2} + \frac{1}{2}\ln(1+e^x). \end{aligned}$$

$\boxed{\text{4 points}}$

Hence the general solution of the inhomogeneous equation is

$$y = c_1 e^x + c_2 e^{-x} - \frac{e^x}{2}\ln(1+e^{-x}) - \frac{1}{2} + \frac{e^{-x}}{2}\ln(1+e^x).$$

with c_1, c_2 being arbitrary constants. $\boxed{\text{2 points}}$