Section A: Calculus

1. (a) Sketch integration region for the double integral

$$I = \int_{x=1}^{x=3} dx \int_{y=0}^{y=\ln x} (x+y) dy$$

By changing the order of integration, evaluate I.

(b) Employ cylindrical coordinates to obtain the volume of the region bounded by the ellipsoid:

$$4(x^2+y^2)+(z-1)^2 = 1,$$

and the planes $z=0$ and $z=3/4$. Recall that the Jacobian determinant for cylindrical coordinates is $|J| = r$.

2. (a) Find and classify the stationary points (maxima, minima and saddle points) of the function

$$f(x,y) = (x-y)\, e^{xy}.$$

(b) Use the method of Lagrange multipliers to find the points on the circle

$$(x-2)^2+(y+1)^2 = 4,$$

which are at maximum and minimum distance from the origin. In each case, compute also the value of the Lagrange multiplier.

3. (a) Consider the following transformation of coordinates

$$x = uv + u^3, \qquad y = uv + v^3,$$

in two dimensional space. Let $f(x,y)$ be a twice differentiable function of x and y. Use the chain rule to obtain $f_{uu} = \frac{\partial^2 f}{\partial u^2}$ and $f_{vv} = \frac{\partial^2 f}{\partial v^2}$ in terms of $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$ and $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$.

(b) [*Not seen*] Let $z = f(x,y)$ be an implicit function of two variables defined by

$$3yz^2 - 3y^2 + 4 = e^{4xz}.$$

Compute $\partial z/\partial x$, $\partial z/\partial y$. Obtain the numerical value of z and its first order partial derivatives at $(x,y) = (1,0)$.

4. (a) Determine the functions $u_1(x)$ and $u_2(x)$ such that $y = c_1 u_1(x) + c_2 u_2(x)$ is the general solution of the homogeneous second-order differential equation

$$y'' - 9y = 0,$$

where c_1, c_2 are arbitrary constants. Compute the Wronskian determinant.

(b) Use the method of variation of parameters to find a particular solution of the inhomogeneous second-order differential equation

$$y'' - 9y = \frac{6}{\sinh(3x)}.$$

Hence determine the general solution of this inhomogeneous equation.