Section A: Calculus

1. (a) Sketch the region of integration in the double integral

$$I = \int_{y=0}^{y=1} dy \int_{x=y}^{x=1} x \sin(2x^3) dx.$$

By changing the order of integration, evaluate I.

(b) The spherical coordinates (r, θ, ϕ) are related to the Cartesian coordinates (x, y, z) by

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi,$$

Obtain the Jacobian determinant of the transformation from Cartesian to spherical coordinates. Hence use spherical coordinates to obtain the volume of the region enclosed by the sphere $x^2 + y^2 + z^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 9$.

2. (a) Find and classify the stationary points (maxima, minima and saddle points) of the function

$$f(x, y) = 2x^3 - 6xy + 3y^2.$$

(b) Compute the Taylor's expansion of the function

$$f(x, y) = xye^{-(x^2+y^2)/2}.$$

around the point $(0, 0)$ including up to second order terms. What can you conclude from the form of the expansion about the nature of the point (0,0)?

Turn over...

3. (a) Let $z = f(x, y)$ be a real function of the two independent variables x and y which is defined implicitly by means of a constraint of the form $G(x, y, z) = 0$. Obtain formulae for the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ in terms of the partial derivatives G_x, G_y, G_z. Give all the details of the derivation. Use these formulae to compute $\partial z/\partial x$ and $\partial z/\partial y$ if

$$G(x, y, z) = \sin(xy) + \cos(yz).$$

(b) Use the method of Lagrange multipliers to find the shortest distance from the point $(3, 0)$ in the xy-plane to the line $y = x$.

4. Determine the functions $u_1(x)$, $u_2(x)$ such that $y(x) = c_1 u_1(x) + c_2 u_2(x)$ is the general solution of the following homogeneous second-order differential equation

$$y'' + y = 0,$$

where c_1, c_2 are arbitrary constants. Show that the Wronskian of u_1, u_2 is nowhere zero.

Use the method of variation of parameters to find a particular solution of the inhomogeneous second-order differential equation

$$y'' + y = e^{-x} + 4 + 2x.$$

Hence determine the general solution of this inhomogeneous equation.

Turn over...

Section B: Linear Algebra

In the following questions, $M(2,2)$ denotes the set of all real-valued 2×2 matrices.

5. (a) Consider the following subset of $\mathbb{R}^3$

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = z\}$$

   i. Show that V is a subspace of $\mathbb{R}^3$.
   ii. Find a basis for V.
   iii. What is the dimension of V?

   (b) Is $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = z^2\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

   (c) Do the following sets form a basis for $\mathbb{R}^3$? If not, determine whether they are linearly independent, a spanning set for $\mathbb{R}^3$, or neither.

   i. $\{(1, 0, 0), (1, 2, -3), (2, 2, -3)\}$.
   ii. $\{(5, 2, 1), (0, 7, 3)\}$.

6. (a) Let V, W be real vector spaces. Define what it means for a map $f : V \to W$ to be linear.

   (b) Define what is meant by the image, the kernel, the rank and the nullity of a linear map and state carefully the Rank-Nullity theorem.

   (c) Consider the map $f : M(2,2) \to \mathbb{R}^2$ given by

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a, b).$$

   i. Prove that f is linear.
   ii. Determine whether f is injective, surjective, both or neither and find a basis for the kernel of f and a basis for the image of f.

   (d) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear map satisfying $f(1, 1) = (1, 2, 3)$ and $f(0, 1) = (0, 1, 5)$. Find $f(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

7. (a) Define what is meant by an eigenvector and an eigenvalue for a real $n \times n$ matrix.

   (b) State carefully the diagonalization theorem for matrices.

   (c) Show that the matrix $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ is diagonalizable and hence find an invertible 3×3 matrix P (and P^{-1}) such that $P^{-1}AP$ is diagonal.

8. (a) Show that

$$\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 2x_2y_2 + 2x_3y_3$$

   for all $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ defines a real inner product on the vector space $\mathbb{R}^3$.

   (b) Define the norm of a vector $\mathbf{x} \in \mathbb{R}^3$ with respect to the above inner product. What is the norm of $(1, 0, 0)$?

   (c) When do we say that two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ are orthogonal (with respect to the above inner product)? Show that $(1, 0, 0)$ and $(0, 1, 1)$ are orthogonal.

   (d) What is an orthonormal set of vectors in $\mathbb{R}^3$ (with respect to the above inner product)? Find $a, b \in \mathbb{R}$ such that
   $\{\mathbf{v_1} = a(1, 0, 0), \mathbf{v_2} = b(0, 1, 1)\}$ form an orthonormal set.

   (e) Using the fact that $\{\mathbf{v_1}, \mathbf{v_2}, (1, 2, 1)\}$ is a basis for $\mathbb{R}^3$, find a vector $\mathbf{v_3}$ such that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthonormal basis for $\mathbb{R}^3$.

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