1. (a) [10 points] Sketch the region bounded by the curves $y = -1$, $y = 1$, $y - x = 3$ and $x = y^2$. Let this region be the integration region R for the integral below. Evaluate the integral.

$$I = \int\int_R (x + y) dx dy.$$

(b) [10 points] The cylindrical coordinates (r, θ, z) are related to the Cartesian co-ordinates (x, y, z) by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Obtain the Jacobian determinant of the transformation from Cartesian to cylindrical coordinates. Hence use cylindrical coordinates to compute the integral

$$V = \int\int\int_R (x^2 + y^2)^2 dx\, dy\, dz$$

on a region R corresponding to a circular cylinder of radius 1 centered at the origin and bounded above by the $z = 1$ plane and below by the $z = 5$ plane.

2. (a) [4 points] Let $f(x, y)$ be a function of two variables and consider the following change of variables:

$$x = t^2 + t + 1 \qquad \text{and} \qquad y = t^2 + 2t + 3,$$

Use the chain rule to obtain f_t in terms of f_x and f_y.
(b) [16 points] Consider now the change of variables:

$$x = s\sin(s + u) \qquad \text{and} \qquad y = u\sin(s - u),$$

Obtain f_s and f_u in terms of f_x and f_y.
Obtain f_{ss} and f_{uu} in terms of s and u if $f(x, y) = xy$.

Note: Simplify your expressions as much as possible. The following formulae may be useful:

$$\begin{aligned}
\cos(s+u)\sin(s-u) + \cos(s-u)\sin(s+u) &= \sin(2s) \\
\cos(s+u)\sin(s-u) - \cos(s-u)\sin(s+u) &= -\sin(2u) \\
2\sin(s-u)\sin(s+u) &= \cos(2u) - \cos(2s)
\end{aligned}$$

Turn over...

3. (a) [8 points] Consider the problem of finding the maximum value of a given function $f(x, y)$ subject to a constraint of the form $\phi(x, y) = 0$. Explain the method of Lagrange multipliers and how it can be applied to solve this kind of problem.

   (b) [12 points] Use the method of Lagrange multipliers to solve the following problem: A music company sells two types of speakers. The profit for selling x speakers of type A and y speakers of type B is given by the function $p(x, y) = x^3 + y^3 - 5xy$. In a given month, the company manufactures at most k speakers, where k is a certain constant. Use the method of Lagrange multipliers to compute the maximum profit that the company can make in a given month.

4. (a) [7 points] Determine the functions $u_1(x)$ and $u_2(x)$ such that $y = c_1u_1(x) + c_2u_2(x)$ is the general solution of the homogeneous second-order differential equation

$$y'' - 2y' + y = 0,$$

   where c_1, c_2 are arbitrary constants. Compute the Wronskian of u_1, u_2.
   (b) [13 points] Use the method of variation of parameters to find a particular solution of the inhomogeneous second-order differential equation

$$y'' - 2y' + y = \frac{e^x}{x}.$$

   Hence determine the general solution of this inhomogeneous equation.

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