Calculus Exam Resit 2007: Solutions

1. (a) The integration region is the triangle in the xy-plane enclosed by the lines $x = 1$, $y = 0$ and $y = x$. $\boxed{3}$

Changing the order of integration we obtain $\boxed{3}$

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=x} x \sin(2x^3) dy.$$

The integral in y gives $\boxed{2}$

$$\int_{y=0}^{y=x} x \sin(2x^3) dy = \left[yx \sin(2x^3) \right]_0^x = x^2 \sin(2x^3).$$

Plugging this result into the second integral we obtain $\boxed{2}$

$$I = \int_{x=0}^{x=1} x^2 \sin(2x^3) dx = \left[-\frac{1}{6} \cos(2x^3) \right]_0^1 = \frac{1 - \cos(2)}{6} = 0.236024...$$

The last integral can be easily carried out using the change of variables $t = x^3$.

(b) The Jacobian of the change of coordinates is simply

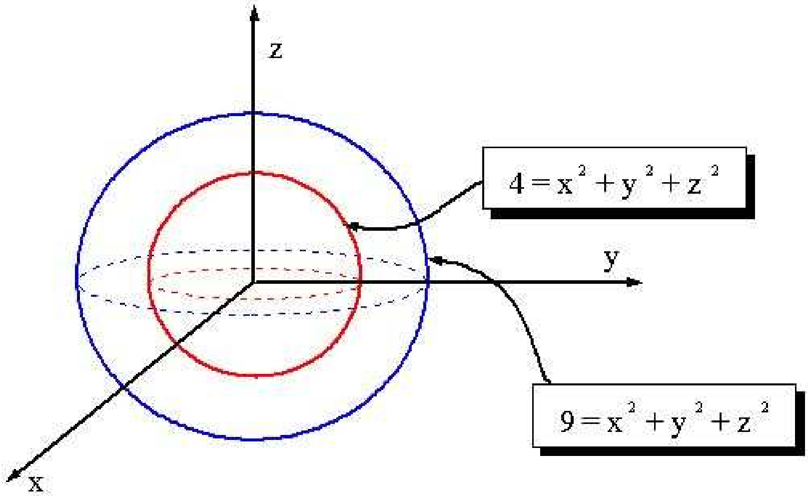
$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta \sin\phi & -r \sin\theta \sin\phi & r \cos\theta \cos\phi \\ \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\ \cos\phi & 0 & -r \sin\phi \end{vmatrix} \\ &= -r^2 \cos^2\theta \sin^3\phi - r^2 \sin^2\theta \cos^2\phi \sin\phi - r^2 \cos^2\theta \cos^2\phi \sin\phi \\ &- r^2 \sin^2\theta \sin^3\phi = -r^2 \sin^3\phi (\cos^2\theta + \sin^2\theta) - r^2 \cos^2\phi \sin\phi (\cos^2\theta + \sin^2\theta) \\ &= -r^2 \sin^3\phi - r^2 \cos^2\phi \sin\phi = -r^2 \sin\phi (\sin^2\phi + \cos^2\phi) = -r^2 \sin\phi. \end{aligned}$$

Therefore the Jacobian of the transformation is

$$dx\, dy\, dz = |J|\, dr\, d\theta\, d\phi = r^2 \sin\phi\, dr\, d\theta\, d\phi.$$

$\boxed{4}$

The integration region for this problem is very easy to sketch. It consists of 2 spheres centered at the origin, one of radius 2 and the other of radius 3. The volume we want to compute is just the volume in between the two spheres (as seen in the picture). An easy way of computing this is simply to compute the volume of each sphere and them subtract the volume of the smallest sphere from that of the largest one.



In spherical coordinates, the equations of the two spheres are simply $r = 2$ and $r = 3$, therefore the integration region for the smallest sphere is

$$R_1 = \{(r, z, \theta) : 0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi\},$$

and for the radius 3 sphere

$$R_2 = \{(r, z, \theta) : 0 \le r \le 3, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi\},$$

and the integrals we want to compute are simply

$$V_1 = \int_{r=0}^{r=2} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi} \sin\phi \, d\phi \qquad \text{and} \qquad V_2 = \int_{r=0}^{r=3} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi} \sin\phi \, d\phi$$

$\boxed{4}$

The various integrals can be carried out separately and give

$$\begin{aligned} \int_{r=0}^{r=2} r^2 dr &= \left[\frac{r^3}{3}\right]_0^2 = \frac{8}{3}, \quad \int_{r=0}^{r=3} r^2 dr = \left[\frac{r^3}{3}\right]_0^3 = 9, \\ \int_{\theta=0}^{\theta=2\pi} d\theta &= 2\pi, \quad \int_{\phi=0}^{\phi=\pi} \sin\phi \, d\phi = [-\cos\phi]_0^\pi = 2. \end{aligned}$$

Therefore

$$V_1 = (2\pi)(8/3)(2) = \frac{32\pi}{3}, \qquad V_2 = (2\pi)(9)(2) = 36\pi,$$

therefore, the required volume is:

$$V = V_2 - V_1 = \frac{76\pi}{3}.$$

$\boxed{2}$

2. (a) First of all we need to find the points at which the first order partial derivatives vanish. These derivatives are

$$f_x = 6x^2 - 6y, \qquad f_y = -6x + 6y.$$

Then

$$f_x = 0 \quad \Leftrightarrow \quad x^2 = y,$$

and

$$f_y = 0 \quad \Leftrightarrow \quad x = y.$$

The two derivatives vanish simultaneously only if $x = y = 0$ or $x = y = 1$. Therefore, we have two points to study: $(0,0)$ and $(1,1)$. The second order partial derivatives are:

$$f_{xx} = 12x, \qquad f_{yy} = 6,$$

$$f_{xy} = f_{yx} = -6.$$

$\boxed{4}$

For the point $(0,0)$ we find:

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0,$$

therefore, the point is a saddle point. $\boxed{3}$

For the point $(1,1)$ we find:

$$f_{xx}f_{yy} - f_{xy}^2 = (12)(6) - 36 = 36 > 0,$$

since $f_{xx}(1,1) = 12 > 0$ the point is a minimum. $\boxed{3}$

(b) The Taylor expansion of a function of two variables $f(x,y)$ around a point (x_0, y_0) up to second order terms is given by $\boxed{2}$

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0, y_0) = (0,0)$ and $\boxed{2}$

$$f_x = y(1-x^2)e^{-(x^2+y^2)/2}, \quad f_y = x(1-y^2)e^{-(x^2+y^2)/2}, \quad f_{xx} = xy(x^2-3)e^{-(x^2+y^2)/2},$$

$$f_{yy} = xy(y^2-3)e^{-(x^2+y^2)/2}, \quad f_{xy} = f_{yx} = (1-x^2)(1-y^2)e^{-(x^2+y^2)/2}.$$

Therefore $\boxed{2}$

$$f(0,0) = 0, \quad f_x(0,0) = 0, \quad f_y(0,0) = 0, \quad f_{xx}(0,0) = 0,$$

$$f_{yy}(0,0) = 0, \quad f_{xy}(0,0) = f_{yx}(0,0) = 1.$$

Hence the Taylor expansion is just $\boxed{2}$

$$f(x,y) = xy.$$

Since $f_x(0,0) = f_y(0,0) = 0$ we know that the point is an stationary point of the function. In addition we have that

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = -1 < 0,$$

and therefore $(0,0)$ is a saddle point of $f(x,y)$. $\boxed{2}$

3. (a) Since $G = 0$ also its total differential $dG = 0$ must vanish. By definition

$$dG = G_x dx + G_y dy + G_z dz = 0,$$

and in addition, z is a function of x and y, therefore its differential is given by

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy.$$

If we substitute dz into dG we obtain the equation

$$dG = 0 = \left(G_x + G_z \frac{\partial z}{\partial x}\right) dx + \left(G_y + G_z \frac{\partial z}{\partial y}\right) dy.$$

Since x and y are independent variables, the equation above implies that each of the factors has to vanish separately, that is

$$G_x + G_z \frac{\partial z}{\partial x} = G_y + G_z \frac{\partial z}{\partial y} = 0.$$

Therefore we obtain,

$$\frac{\partial z}{\partial x} = -\frac{G_x}{G_z}, \qquad \frac{\partial z}{\partial y} = -\frac{G_y}{G_z}.$$

$\boxed{6}$

Employing now these formulae for $G(x,y,z) = \sin(xy) + \cos(yz)$ we obtain

$$\frac{\partial z}{\partial x} = \frac{\cos(xy)}{\sin(yz)}, \qquad \frac{\partial z}{\partial y} = \frac{x\cos(xy)}{y\sin(yz)}.$$

$\boxed{4}$

(b) The function we need to minimize in this case is the distance from a point in the line (which we want to determine) to the point $(3,0)$. The distance square function is given by:

$$f(x,y) = (x-3)^2 + y^2.$$

The point (x,y) must be a point on the line $y = x$, therefore our constraint is $\phi(x,y) = y - x = 0$.

The partial derivatives of f and ϕ are:

$$f_x = 2(x-3), \quad f_y = 2y, \quad \phi_x = -1, \quad \phi_y = 1.$$

Therefore, the system of equations which we need to solve is:

$$y - x = 0,$$

$$2(x-3)-\lambda=0,$$
$$2y+\lambda=0.$$

From the two last equations, we obtain: $\boxed{5}$

$$\lambda=-2y=2(x-3)$$

and using the first equation to substitute $y=x$ the equation above becomes:

$$2(2x-3)=0,$$

which has only one solution $x=3/2$. Therefore there is only one point which solves the problem, that is $(x,y)=(3/2,3/2)$ which corresponds to $\lambda=-3$. The square of the distance from this point to the line $y=x$ is,

$$f(3/2,3/2)=9/2.$$

$\boxed{5}$

4. To obtain the general solution of the homogeneous equation we try solutions of the type $y=ce^{mx}$. Substituting this solution into the equation we obtain the condition $\boxed{2}$

$$m^2+1=0\Rightarrow m=\pm i.$$

This means that the general solution of the homogeneous equation is of the form $\boxed{2}$

$$y=c_1\sin(x)+c_2\cos(x),$$

therefore we identify

$$u_1(x)=\sin(x),\qquad u_2(x)=\cos(x).$$

For the second part of the problem we will need the Wronskian of these solutions which is $\boxed{3}$

$$\begin{aligned} W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} \\ &= -\sin^2 x-\cos^2 x=-1. \end{aligned}$$

Therefore the Wronskian is indeed nowhere zero.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form $\boxed{3}$

$$y=v_1(x)u_1(x)+v_2(x)u_2(x),$$

with

$$v_1(x)=-\int u_2(x)\frac{R(x)}{W(x)}dx\qquad\text{and}\qquad v_2(x)=\int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x)=e^{-x}+4+2x,\qquad W(x)=-1,$$

therefore $\boxed{4}$

$$v_1(x) = \int (e^{-x} + 4 + 2x)\cos(x)dx = \int e^{-x}\cos(x)dx + \int 4\cos(x)dx + 2\int x\cos(x)dx,$$

Integrating by parts twice, we find

$$\int e^{-x}\cos(x)dx = \frac{e^{-x}(\sin(x) - \cos(x))}{2}.$$

Integrating by parts once we obtain

$$2\int x\cos(x)dx = 2(\cos(x) + x\sin(x)).$$

Therefore

$$\begin{aligned} v_1(x) &= \frac{e^{-x}(\sin(x) - \cos(x))}{2} + 2(\cos(x) + x\sin(x)) + 4\sin(x) \\ &= \frac{\sin(x)}{2}(e^{-x} + 4x + 8) + \frac{\cos(x)}{2}(4 - e^{-x}). \end{aligned}$$

$\boxed{4}$

Similarly,

$$v_2(x) = -\int (e^{-x} + 4 + 2x)\sin(x)dx = \frac{\sin(x)}{2}(e^{-x} - 4) + \frac{\cos(x)}{2}(e^{-x} + 8 + 4x).$$

Hence the general solution of the inhomogeneous equation (after simplifying several terms) is $\boxed{2}$

$$y = c_1\sin(x) + c_2\cos(x) + \frac{1}{2}(e^{-x} + 8 + 4x).$$

with c_1, c_2 being arbitrary constants.

Linear Algebra Exam Resit 2007: Solutions

5. (a) i. Need to check that conditions (S1)-(S3) are satisfied.
(S1) $(0,0,0) \in V$ as $0 + 0 = 0$.
(S2) If $(x,y,z), (x',y',z') \in V$ (i.e. $x+y=z$ and $x'+y'=z'$) then $(x,y,z) + (x',y',z') = (x+x', y+y', z+z') \in V$ as

$$(x+x') + (y+y') = (x+y) + (x'+y') = z + z'$$

(S3) If $(x,y,z) \in V$ (i.e. $x+y=z$) and $\lambda \in \mathbb{R}$ then $\lambda(x,y,z) = (\lambda x, \lambda y, \lambda z) \in V$ as

$$\lambda x + \lambda y = \lambda(x+y) = \lambda z.$$

ii. V can be written as $\{(x, y, x + y) \mid x, y \in \mathbb{R}\}$. A basis for V is given by $\{(1, 0, 1), (0, 1, 1)\}$. It is a spanning set as

$$(x, y, x + y) = x(1, 0, 1) + y(0, 1, 1) \qquad \forall x, y \in \mathbb{R}$$

and it is also linearly independent as these two vectors are not multiple of each other.

iii. Using (ii) we see that the dimension of V is 2.

[9]

(b) W is not a subspace of $\mathbb{R}^3$ as, for example, condition (S3) fails. Take $(1, 0, 1) \in W$ and $\lambda = 2$ then $\lambda(1, 0, 1) = (2, 0, 2) \notin W$ as $2 + 0 \neq 2^2$.

[3]

(c) i. Is this set linearly independent? No, as

$$(2, 2, -3) = (1, 0, 0) + (1, 2, -3).$$

Is it a spanning set for $\mathbb{R}^3$? No as this set contains (at most) two linearly independent vectors, which is not enough to span $\mathbb{R}^3$. Alternatively, write

$$(a, b, c) = \lambda_1(1, 0, 0) + \lambda_2(1, 2, -3) + \lambda_3(2, 2, -3).$$

This is equivalent to

$$\begin{cases} a = \lambda_1 + \lambda_2 + 2\lambda_3 \\ b = 2\lambda_2 + 2\lambda_3 \\ c = -3\lambda_2 - 3\lambda_3 \end{cases}$$

and implies $\frac{b}{2} = -\frac{c}{3}$. So in particular, $(0, 1, 1)$ is not in the span of these three vectors.

As it is not linearly independent (and not spanning), it is not a basis for $\mathbb{R}^3$.

ii. Is it linearly independent? Write

$$\lambda_1(5, 2, 1) + \lambda_2(0, 7, 3) = (0, 0, 0)$$

This is equivalent to the system of equations

$$\begin{cases} 5\lambda_1 = 0 \\ 2\lambda_1 + 7\lambda_2 = 0 \\ \lambda_1 + 3\lambda_2 = 0 \end{cases}$$

The only solution is $\lambda_1 = \lambda_2 = 0$. Thus this set is linearly independent.

Is it a spanning set for $\mathbb{R}^3$? No as we need at least 3 vectors to span $\mathbb{R}^3$.

As it is not spanning, it is not a basis.

[8]

6. (a) A map $f : V \to W$ is linear if and only if the following two conditions are satisfied:

$$(i)\ f(u+v) = f(u) + f(v) \qquad \forall u, v \in V$$

$$(ii)\ f(\lambda v) = \lambda f(v) \qquad \forall v \in V, \lambda \in \mathbb{R}$$

[2]

(b) Let $f : V \to W$ be a linear map from a vector space V to a vector space W. The image of f is defined by $\operatorname{Im} f = \{w \in W \,:\, w = f(v) \text{ for some } v \in V\}$. The kernel of f is defined by $\operatorname{Ker} f = \{v \in V \,:\, f(v) = 0\}$. The rank of f is the dimension of the image of f. The nullity of f is the dimension of the kernel of f.

The rank-nullity theorem says that if V is finite dimensional then we have

$$\dim V = \operatorname{rank} f + \operatorname{nullity} f.$$

[4]

(c) i.

$$\begin{aligned} f(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}) &= f(\begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}) \\ &= (a+a', b+b') \\ &= (a,b) + (a',b') \\ &= f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) + f(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}). \end{aligned}$$

$$\begin{aligned} f(\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}) &= f(\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}) \\ &= (\lambda a, \lambda b) = \lambda(a,b) \\ &= \lambda f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}). \end{aligned}$$

[4]

ii.

$$\begin{aligned} \operatorname{Ker} f &= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid (a,b) = (0,0)\} \\ &= \{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \mid c, d \in \mathbb{R}\} \neq \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \end{aligned}$$

So f is not injective.

A basis for $\operatorname{Ker} f$ is given by $\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ (clearly spanning and

linearly independent).
The Rank-Nullity theorem says

$$\dim M(2,2) = 2 + \operatorname{rank} f$$

and as $\dim M(2,2) = 4$ we get that $\dim \operatorname{Im} f = 2$. As $\operatorname{Im} f$ is a subspace of $\mathbb{R}^2$ of dimension 2, we must have $\operatorname{Im} f = \mathbb{R}^2$. So f is surjective and $\operatorname{Im} f$ has a basis given by $\{(1,0),(0,1)\}$.

[6]

(d) $(x,y) = x(1,1) + (y-x)(0,1)$ so we must have

$$\begin{aligned} f(x,y) &= xf(1,1) + (y-x)f(0,1) \\ &= x(1,2,3) + (y-x)(0,1,5) \\ &= (x,2x,3x) + (0,y-x,5y-5x) \\ &= (x,x+y,5y-2x). \end{aligned}$$

[4]

7. (a) An eigenvector for an $n \times n$ matrix A is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$. An eigenvalue for A is a real number λ such that there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \lambda\mathbf{x}$.

[3]

(b) Suppose A has n linearly independent eigenvectors then there exists an invertible matrix P (whose columns are these eigenvectors) such that $P^{-1}AP$ is diagonal.

[3]

(c)

$$\det \begin{pmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix} = 0$$

This gives

$$\lambda^2(6-\lambda)^2 = 0.$$

Thus the eigenvalues of A are 0 and 6.

[3]

When $\lambda = 0$ we have

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the eigenspace is given by

$$s_A(0) = \{(x,y,-x-y) \,:\, x,y \in \mathbb{R}\}$$

with basis given by $\{(1,0,-1),(0,1,-1)\}$ (clearly spanning and linearly independent).

[3]

When $\lambda = 6$ we have

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the eigenspace is given by

$$s_A(-1) = \{(x,x,x) \, : \, x \in \mathbb{R}\}$$

with basis given by $\{(1,1,1)\}$ (clearly spanning and linearly independent).

[3]

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

[5]

8. (a) (i) For all $x, y \in \mathbb{R}^3$ we have

$$\begin{aligned} \langle x, y \rangle &= 2x_1y_1 + 2x_2y_2 + 2x_3y_3 \\ &= 2y_1x_1 + 2y_2x_2 + 2y_3x_3 \\ &= \langle y, x \rangle. \end{aligned}$$

(ii) For all $x, y, z \in \mathbb{R}^3$ we have

$$\begin{aligned} \langle x + y, z \rangle &= 2(x_1 + y_1)z_1 + 2(x_2 + y_2)z_2 + 2(x_3 + y_3)z_3 \\ &= (2x_1z_1 + 2x_2z_2 + 2x_3z_3) + (2y_1z_1 + 2y_2z_2 + 2y_3z_3) \\ &= \langle x, z \rangle + \langle y, z \rangle. \end{aligned}$$

(iii) For all $x, y \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \langle \lambda x, y \rangle &= 2(\lambda x_1)y_1 + 2(\lambda x_2)y_2 + 2(\lambda x_3)y_3 \\ &= \lambda(2x_1y_1 + 2x_2y_2 + 2x_3y_3) = \lambda\langle x, y \rangle. \end{aligned}$$

(iv) For all $x \in \mathbb{R}^3$ we have $\langle x, x \rangle = 2x_1^2 + 2x_2^2 + 2x_3^2 \geq 0$ and we have equality if and only if $x = (0,0,0)$.

[6]

(b) The norm of a vector x is given by $||x|| = \sqrt{2x_1^2 + 2x_2^2 + 2x_3^2}$.
$||(1,0,0)|| = \sqrt{2}$. [2]

(c) Two vectors x and y are orthogonal if and only if $\langle x, y\rangle = 0$. The two vectors $(1,0,0)$ and $(0,1,1)$ are orthogonal as

$$\langle (1,0,0),(0,1,1)\rangle = 2.1.0 + 2.0.1 + 2.0.1 = 0.$$

[2]

(d) A set of vectors is orthonormal if they are pairwise orthogonal and they all have norm 1. The vectors $(1,0,0)$ and $(0,1,1)$ are already orthogonal so all that is left to do is to divide them by their respective norm.

$$||(1,0,0)|| = \sqrt{2} \quad \text{and} \quad ||(0,1,1)|| = 2$$

so we get

$$\mathbf{v_1} = \frac{1}{\sqrt{2}}(1,0,0) = (\frac{1}{\sqrt{2}},0,0)$$

and

$$\mathbf{v_2} = \frac{1}{2}(0,1,1) = (0,\frac{1}{2},\frac{1}{2})$$

i.e. $a = \frac{1}{\sqrt{2}}$ and $b = \frac{1}{2}$.

[4]

(e) First define

$$\begin{aligned}
\mathbf{w_3} &= (1,2,1) - \langle (1,2,1),\mathbf{v_1}\rangle\mathbf{v_1} - \langle (1,2,1),\mathbf{v_2}\rangle\mathbf{v_2} \\
&= (1,2,1) - \frac{2}{\sqrt{2}}(\frac{1}{\sqrt{2}},0,0) - 3(0,\frac{1}{2},\frac{1}{2}) \\
&= (0,\frac{1}{2},-\frac{1}{2}).
\end{aligned}$$

Now

$$||(0,\frac{1}{2},-\frac{1}{2})|| = 1$$

so we have $\mathbf{v_3} = \mathbf{w_3} = (0,\frac{1}{2},-\frac{1}{2})$.

[6]