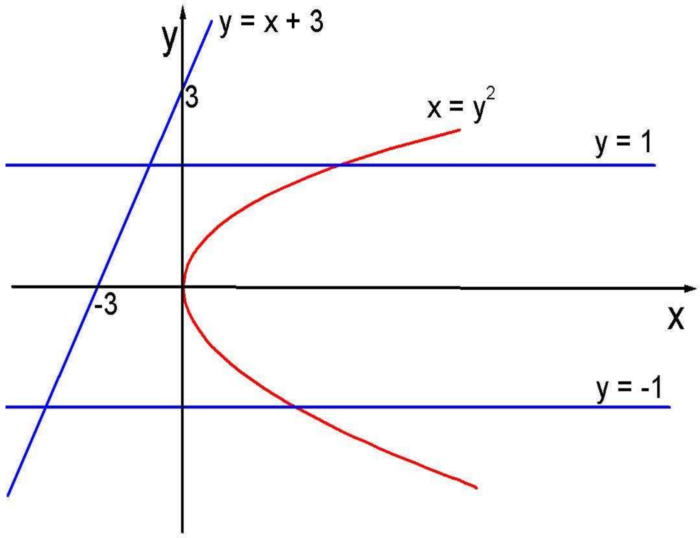
Calculus 2008: Solutions

1. (a) The integration region is shown in the figure below:



3 points

The integral is simply

$$\int_{y=-1}^{y=1}\int_{x=y-3}^{x=y^2}(x+y)dxdy = \int_{y=-1}^{y=1}\left[\frac{x^2}{2}+yx\right]_{x=y-3}^{x=y^2} dy$$

$$= \int_{y=-1}^{y=1}\left(\frac{y^4}{2}+y^3-\frac{(y-3)^2}{2}-y(y-3)\right)dy = \left[\frac{y^5}{10}+\frac{y^4}{4}-\frac{(y-3)^3}{6}-\frac{y^3}{3}+\frac{3y^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{10}+\frac{1}{4}-\frac{(-2)^3}{6}-\frac{1}{3}+\frac{3}{2}+\frac{1}{10}-\frac{1}{4}+\frac{(-4)^3}{6}-\frac{1}{3}-\frac{3}{2} = \frac{1}{5}-\frac{2}{3}+\frac{4}{3}-\frac{32}{3} = \frac{1}{5}-10 = -\frac{49}{5}.$$

7 points

(b) The Jacobian of the change of coordinates is simply 2

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, the element of volume which we need to use for the integral is 1

$$dx\,dy\,dz = |J|\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz.$$

To compute the integral we have first to express the integrand in terms of the new variables, that is 1

$$(x^2+y^2)^2 = (r^2)^2 = r^4.$$

The next step is to describe the region of integration in terms of the new variables. The integration region for this problem is very easy to sketch. We have a radius 1 circular cylinder centered at the origin extending between the $z=1$ and $z=5$ planes.

In cylindrical coordinates, the integration region is simply 2

$$R = \{(r,z,\theta) : 0 \le r \le 1, \quad 1 \le z \le 5, \quad 0 \le \theta \le 2\pi\},$$

and the integral we want to compute is therefore 1

$$V = \int_{r=0}^{r=1} r^5 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{z=1}^{z=5} dz.$$

The various integrals can be carried out separately and give 2

$$\int_{r=0}^{r=1} r^5 dr \quad = \quad \left[\frac{r^6}{6}\right]_0^1 = \frac{1}{6}, \quad \int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi, \quad \int_{z=1}^{z=5} dz = 5-1 = 4.$$

Therefore 1

$$V = (2\pi)(1/6)(4) = \frac{4\pi}{3}.$$

2. (a) Here we just have to use the chain rule

$$f_t = \frac{\partial x}{\partial t} f_x + \frac{\partial y}{\partial t} f_y = (2t+1)f_x + (2t+2)f_y.$$

(b) Here we use the chain rule again and find 4

$$f_s = \frac{\partial x}{\partial s} f_x + \frac{\partial y}{\partial s} f_y = (\sin(s+u) + s\cos(s+u))f_x + u\cos(s-u)f_y,$$

and 3

$$f_u = \frac{\partial x}{\partial u} f_x + \frac{\partial y}{\partial u} f_y = (s\cos(s+u))f_x + (\sin(s-u) - u\cos(s-u))f_y.$$

3

Next, we want to obtain f_{ss}, f_{uu} for the function $f(x,y) = xy$. The simplest way to do this is to compute first f_s and f_u for this particular function. Since

$$f_x = y = u\sin(s-u), \qquad f_y = x = s\sin(s+u), \qquad f_{xx} = f_{yy} = 0, \qquad f_{xy} = f_{yx} = 1,$$

we find

$$f_s = (\sin(s+u) + s\cos(s+u))u\sin(s-u) + su\cos(s-u)\sin(s+u),$$

and

$$f_u = su\cos(s+u)\sin(s-u) + (\sin(s-u) - u\cos(s-u))s\sin(s+u),$$

which can be simplified by using the formulae given in the exam to

$$f_s = -\frac{u}{2}\left(\cos(2s) - \cos(2u)\right) + su\sin(2s),$$

3

$$f_u = -\frac{s}{2}\left(\cos(2\,s) - \cos(2\,u)\right) - su\sin(2u).$$

From these expressions it is then easy to find 3

$$f_{ss} = 2u(s\cos(2s) + \sin(2s)), \qquad f_{uu} = -2s(u\cos(2u) + \sin(2u)).$$

4

3. (a) Let the point (x, y) be a maximum of $f(x, y)$, then

$$df = f_x dx + f_y dy = 0,$$

since the first order partial derivatives always vanish at a maximum point. 1.5 points

Since $\phi(x, y) = 0$, then it follows trivially that

$$d\phi = \phi_x dx + \phi_y dy = 0,$$

and therefore we can also write that 1.5 points

$$d(f + \lambda\phi) = df + \lambda d\phi = 0,$$

where λ is an arbitrary constant which we call **Lagrange's multiplier**. 1 points

The previous equation is equivalent to

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right) dy = 0.$$

2 points

Since λ is arbitrary we can choose it so that

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0.$$

which implies

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0.$$

Therefore, we have now a set of 3 equations, for 3 unknowns namely

$$\phi(x, y) = f_x + \lambda\phi_x = f_y + \lambda\phi_y = 0.$$

2 points

(b) The function we want to maximize is the profit $p(x, y)$ and the constraint is that the total number of speakers is at most k. We can express that as $x + y = k$ and therefore our constraint is the function $\phi(x, y) = x + y - k = 0$. The system of equations which we need to solve is:

$$\begin{aligned} f_x + \lambda\phi_x &= 0 \qquad \Rightarrow \quad 3x^2 - 5y + \lambda = 0 \\ f_y + \lambda\phi_y &= 0 \qquad \Rightarrow \quad 3y^2 - 5x + \lambda = 0 \\ \\ x + y - k &= 0, \end{aligned}$$

From the 1st two equations we get that **4 points**

$$3x^2 - 5y = 3y^2 - 5x \qquad \Rightarrow \quad (x - y)(3(x + y) + 5) = 0,$$

which has solutions $x = y$ or $x + y = -5/3$. Now we have to check whether or not these two solutions make sense. In fact, the second solutions is clearly not possible since x and y are the numbers of speakers produced by the company in a month and the sum of these numbers can never be negative! Thus we are left with only one solution, $x = y$. **4 points**

Substituting it in the constraint we get,

$$x = y = k/2,$$

and $\lambda = \frac{k(10-3k)}{4}$. For these values of x and y the profit becomes:

$$p(k/2, k/2) = \frac{k^2(k - 5)}{4},$$

and this is the solution to our problem. Notice that the problem has a meaningful solution only if $k > 5$. **4 points**

4. (a) We try, as usual, solutions of the type $y = e^{mx}$. Putting this into the homogeneous equation we obtain

$$m^2 - 2m + 1 = 0 \quad \Leftrightarrow \quad m = 1.$$

This gives us only one solution! So, in order to find the other independent solution, we need to try something different. For example, take $y = xe^{ax}$. If we put this into our equation we obtain:

$$(a + a(1 + ax)) - 2(1 + ax) + x = 0 \quad \Leftrightarrow \quad a = 1.$$

Therefore $u_1(x) = e^x$ and $u_2(x) = xe^x$. **4 points**

The Wronskian is

$$W(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x) = e^x(e^x + xe^x) - e^x xe^x = e^{2x}.$$

(b) A particular solution of the inhomogeneous equation is given by **3 points**

$$y_p = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx, \qquad v_2(x) = -\int u_1(x)\frac{R(x)}{W(x)}dx.$$

Here $R(x) = \frac{e^x}{x}$ which gives | 3 points |

$$v_1(x) = -\int xe^x \frac{e^x}{xe^{2x}}dx = -\int dx = -x,$$

and | 4 points |

$$v_2(x) = \int e^x \frac{e^x}{xe^{2x}}dx = \int \frac{dx}{x} = \ln x.$$

Therefore, the general solution of the inhomogeneous equation is given by | 4 points |

$$y = (c_1 + c_2 x)e^x - xe^x + xe^x \ln x = e^x[(c_1 + c_2 x) - x(1 - \ln x)],$$

for c_1, c_2 arbitrary constants. | 2 points |