2.3 Differentiation of functions of several real variables

In this section we will begin the process of extending the concepts and techniques of single-variable calculus to functions of more than one variable. It is convenient to begin by considering the rate of change of such functions with respect to one variable at a time. This is what we will call **first-order partial derivative**. We will denote the first-order partial derivative with respect to the variable x as

$$\frac{\partial}{\partial x}, \tag{2.39}$$

in order to distinguish it from the total derivative

$$\frac{d}{dx}, \tag{2.40}$$

which we used for functions of one variable. Thus, a function of n variables has n first-order partial derivatives, one with respect to each of its independent variables

$$\frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_i, \ldots, x_n) \quad \text{for} \quad i = 1, \ldots, n. \tag{2.41}$$

Let us now provide a more rigorous definition for functions of just two variables.

2.3.1 Definition of partial derivative for a function of two variables

Definition: The first partial derivatives of the function $f(x, y)$ with respect to the variables x and y are given by

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \tag{2.42}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}. \tag{2.43}$$

Note: Notice that $\partial f/\partial x$ is nothing but the standard first derivative of $f(x, y)$ when considered as a function of x only, regarding y as a constant parameter. Similarly $\partial f/\partial y$ is the standard first derivative of $f(x, y)$ when considered as a function of y only, regarding x as a constant parameter.

Example: Let

$$f(x, y) = x^2 \cos y + 2xy, \tag{2.44}$$

then, according to the definition above, the partial derivatives of $f(x, y)$ with respect to x and y are

$$\frac{\partial f}{\partial x} = \cos y \frac{dx^2}{dx} + 2y \frac{dx}{dx} = 2x \cos y + 2y, \tag{2.45}$$

$$\frac{\partial f}{\partial y} = x^2 \frac{d \cos y}{dy} + 2x \frac{dy}{dy} = -x^2 \sin y + 2x. \tag{2.46}$$

Geometric interpretation: Partial derivatives of functions of two variables admit a similar geometrical interpretation as for functions of one variable. For a function of one variable $f(x)$, the first derivative with respect to x is defined as

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{2.47}$$

and geometrically it measures the slope of the curve $f(x)$ at the point x. This is illustrated in figure 5.

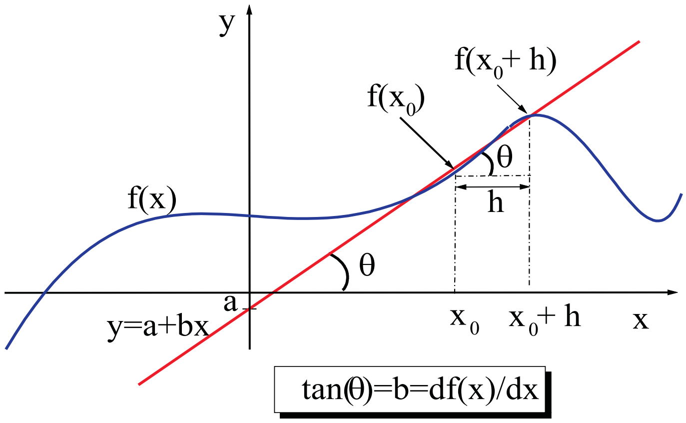

Figure 5: *Geometrical picture of df/dx at the point $x = x_0$. Applying the definition (2.47), the derivative is nothing but the slope of the line $y = a + bx$ which is tangent to $f(x)$ at the point x_0, namely $b = \frac{df}{dx}|_{x_0}$.*

Recalling the definition we gave in the note above, we can now interpret geometrically the first-order partial derivatives of a function of two variables in a completely analog fashion:

Geometrical definition of f_x and f_y: The partial derivative $\partial f/\partial x$ at a certain point (x_0, y_0) is nothing but the slope of the curve of intersection of the function $f(x, y)$ and the vertical plane $y = y_0$ at $x = x_0$. Likewise, the partial derivative $\partial f/\partial y$ at a certain point (x_0, y_0) is nothing but the slope of the curve of intersection of the function $f(x, y)$ and the horizontal plane $x = x_0$ at $y = y_0$. Graphically:

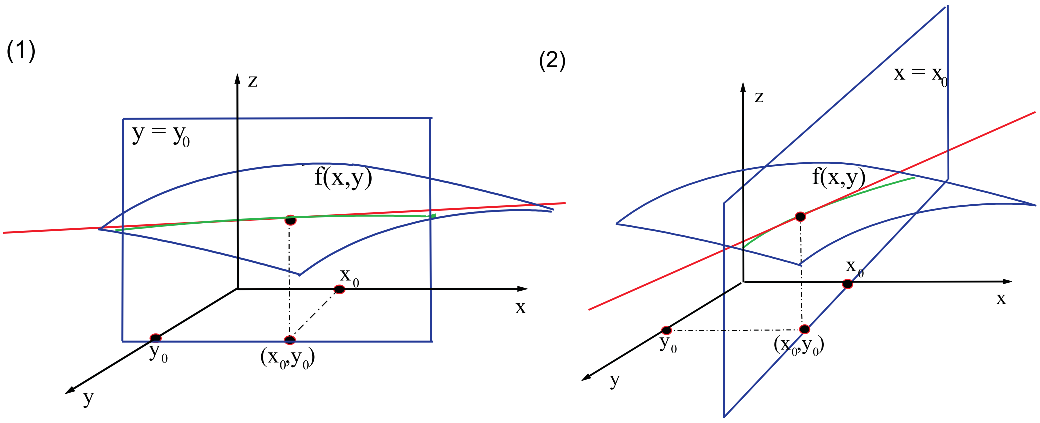

Figure 6: *Geometrical picture of $\partial f/\partial x$ (1) and $\partial f/\partial y$ (2) at the point (x_0, y_0). In (1) $\partial f/\partial x$ is the slope of the red line which is tangent to the green curve resulting from the intersection of $f(x, y)$ and the plane $y = y_0$. In (2) $\partial f/\partial y$ can be identified in a similar fashion.*

The geometric interpretation of partial derivatives is also rather well explained in:
http://www.math.uri.edu/Center/workht/calc3/tangent1.html
and can be visualized in
http://www-math.mit.edu/18.013A/HTML/tools/tools22.html

Notation: From now on we will employ the following shorter notation for the partial derivatives of $f(x, y)$

$$\frac{\partial f}{\partial x} = f_x, \qquad \frac{\partial f}{\partial y} = f_y. \tag{2.48}$$

We will denote by $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ the partial derivatives at the point (x_0, y_0).

<u>**Definition:**</u> Given a function $f(x, y)$, the function is said to be **differentiable** if f_x and f_y exist. If the function is differentiable its first-order derivatives can be differentiated again and we can define the **second-order partial derivatives** of $f(x, y)$ as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \tag{2.49}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}, \tag{2.50}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}, \tag{2.51}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}. \tag{2.52}$$

<u>**Note:**</u> Notice that f_{xy} and f_{yx} are in principle different functions. Using the definitions (2.42)-(2.43) twice we can see that the second derivatives f_{xy} and f_{yx} are given by the double limits

$$\begin{aligned} f_{xy} &= \lim_{p\to 0} \frac{f_y(x+p, y) - f_y(x, y)}{p} \qquad (2.53)\\ &= \lim_{p\to 0}\left[\lim_{h\to 0} \frac{f(x+p, y+h) - f(x+p, y) - f(x, y+h) + f(x, y)}{hp}\right], \\ f_{yx} &= \lim_{h\to 0} \frac{f_x(x, y+h) - f_x(x, y)}{h} \qquad (2.54)\\ &= \lim_{h\to 0}\left[\lim_{p\to 0} \frac{f(x+p, y+h) - f(x+p, y) - f(x, y+h) + f(x, y)}{hp}\right]. \end{aligned}$$

Therefore, the only difference between f_{xy} and f_{yx} is the order in which the limits are taken. **It is not guaranteed that the limits commute**.

Example 1: Let us compute the first- and second-order partial derivatives of the function

$$f(x, y) = e^{xy} + \ln(x^2 + y). \tag{2.55}$$

We start with the 1st derivatives,

$$f_x = ye^{xy} + \frac{2x}{x^2 + y}, \tag{2.56}$$

$$f_y = xe^{xy} + \frac{1}{x^2 + y}. \tag{2.57}$$

Now we can compute the 2nd derivatives,

$$f_{xy} = \frac{\partial f_y}{\partial x} = e^{xy} + xye^{xy} - \frac{2x}{(x^2 + y)^2}, \tag{2.58}$$

$$f_{yx} = \frac{\partial f_x}{\partial y} = e^{xy} + xye^{xy} - \frac{2x}{(x^2 + y)^2}. \tag{2.59}$$

So, in this case $f_{xy} = f_{yx}$.

Example 2: As we said at the beginning of this section, all definitions for functions of two variables extend easily to functions of 3 or more variables. In this example let us consider the function of three variables

$$g(x, y, z) = e^{x-2y+3z}, \tag{2.60}$$

and compute its 1st and 2nd order partial derivatives. In this case we have 3 1st order derivatives

$$g_x = e^{x-2y+3z}, \qquad g_y = -2e^{x-2y+3z}, \qquad g_z = 3e^{x-2y+3z}. \tag{2.61}$$

Now we have in total 9 possible different 2nd order derivatives,

$$g_{xx} = e^{x-2y+3z}, \qquad g_{yx} = -2e^{x-2y+3z}, \qquad g_{zx} = 3e^{x-2y+3z}, \tag{2.62}$$

$$g_{xy} = -2e^{x-2y+3z}, \qquad g_{yy} = 4e^{x-2y+3z}, \qquad g_{zy} = -6e^{x-2y+3z}, \tag{2.63}$$

$$g_{xz} = 3e^{x-2y+3z}, \qquad g_{yz} = -6e^{x-2y+3z}, \qquad g_{zz} = 9e^{x-2y+3z}. \tag{2.64}$$

Definition: Consider a function of two variables $f(x, y)$ and let f,f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx} exist and be **continuous** in a neighbourhood of a point (x_0, y_0). Then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0). \tag{2.65}$$

A nice proof of this theorem is given in chapter 13 of the book "Calculus" by R. Adams. The key idea is that the continuity of all 1st and 2nd order derivatives of f allows us to proof that the order of the limits in (2.53) and (2.54) is irrelevant for the final result.