

2.3.4 Functions of several variables defined implicitly

As usual let us start by looking at the easiest example, namely functions of one variable.

Definition: We say that the function $y = f(x)$ is defined implicitly if y and x are related by an equation of the type

$$\Phi(x, y) = 0, \quad (2.115)$$

and there is no possibility of obtaining $y = f(x)$ explicitly from the constraint (2.115).

Note: Notice however that the equation (2.115) allows us to obtain the value of y for a given value of x (at least numerically), even though it does not allow us to know $f(x)$ for arbitrary x .

It is easier to understand this definition with some examples:

Example 1: Let

$$\Phi(x, y = f(x)) = \log(x + y) - \sin(x + y) = 0. \quad (2.116)$$

The constraint $F(x, y) = 0$ gives us a relation between x and y , however we are not able to obtain y as a function of x from this equation. Therefore the function $y = f(x)$ is defined implicitly through the constraint $\Phi(x, y) = 0$.

Suppose now that we would like to know the value of y for $x = 0$. In this case we have

$$\Phi(0, y) = 0 = \log(y) - \sin(y) \quad \Rightarrow \quad \log(y) = \sin(y), \quad (2.117)$$

and this equation can be solved numerically by finding the points at which the curves $\log(y)$ and $\sin(y)$ meet. The solution is approximately $y = f(0) \simeq 2.22$.

Example 2: Let

$$G(x, y = f(x)) = e^{xy} - x^2 y \log(y) - 1 = 0. \quad (2.118)$$

Again $y = f(x)$ is an implicit function of x , since there is no way from the constraint $G(x, y) = 0$ to obtain y as function of x explicitly.

We can however obtain the value of y for fixed values of x , for example at $x = 1$ we have

$$e^y = y \log(y) + 1. \quad (2.119)$$

The approximate numerical solution of this equation is $y = f(1) \simeq 1.16$.

The definition for functions of one variable extends easily to functions of several variables. For example, for functions of two variables $z = f(x, y)$ we have:

Definition: A function of two variables $z = f(x, y)$ is said to be an implicit function of the independent variables x and y if z, x and y are related to each other by a constraint of the type

$$\Phi(x, y, z) = 0, \quad (2.120)$$

from which z can not be expressed explicitly in terms of x and y . As for the case of functions of one variable, it will still be possible to obtain z from the condition above for fixed values of (x, y) .

Example: Consider the function $z = f(x, y)$ defined by

$$x^3 + 2x^2yz + \sin z - 1 = 0, \quad (2.121)$$

It is not possible to obtain $z = f(x, y)$ from the equation above for generic values of x, y . However we can solve it for given values of x and y . For example, take $x = y = 1$

$$2z + \sin(z) = 0, \quad (2.122)$$

which has solution $z = f(1, 1) = 0$.

Partial derivatives of implicit functions: The main problem we want to address in this section is the following: **Given a function of several variables defined implicitly, how can we obtain the partial derivatives of this function?** Let us consider an implicit function of two variables $z = f(x, y)$ and assume the existence of a constraint

$$\Phi(x, y, z) = 0, \quad (2.123)$$

which relates the function z to the two independent variables x and y . Since $\Phi = 0$ it is clear that also its total differential $d\Phi = 0$ must vanish. However the total differential is by definition

$$d\Phi = \left(\frac{\partial\Phi}{\partial x}\right) dx + \left(\frac{\partial\Phi}{\partial y}\right) dy + \left(\frac{\partial\Phi}{\partial z}\right) dz = 0, \quad (2.124)$$

and in addition, z is a function of x and y , therefore its differential is given by

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy. \quad (2.125)$$

If we substitute (2.125) into (2.124) we obtain the equation

$$d\Phi = 0 = \left(\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial z} \frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial\Phi}{\partial y} + \frac{\partial\Phi}{\partial z} \frac{\partial z}{\partial y}\right) dy. \quad (2.126)$$

Since x and y are independent variables, equation (2.126) implies that each of the factors has to vanish separately, that is

$$\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial\Phi}{\partial y} + \frac{\partial\Phi}{\partial z} \frac{\partial z}{\partial y} = 0. \quad (2.127)$$

Therefore we obtain,

$$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial\Phi}{\partial x}\right)}{\left(\frac{\partial\Phi}{\partial z}\right)} = -\frac{\Phi_x}{\Phi_z}, \quad (2.128)$$

$$\frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial\Phi}{\partial y}\right)}{\left(\frac{\partial\Phi}{\partial z}\right)} = -\frac{\Phi_y}{\Phi_z}. \quad (2.129)$$

Example 1: Consider again the function of the previous example

$$\Phi(x, y, z) = x^3 + 2x^2yz + \sin z - 1 = 0, \quad (2.130)$$

where z is an implicit function of the independent variables x and y . Obtain the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$.

Here we must only apply the equations (2.128) and (2.129). For that we need the partial derivatives

$$\Phi_x = 3x^2 + 4xyz, \quad \Phi_y = 2x^2z \quad \text{and} \quad \Phi_z = 2x^2y + \cos z, \quad (2.131)$$

and so we obtain

$$\frac{\partial z}{\partial x} = -\frac{\Phi_x}{\Phi_z} = -\frac{3x^2 + 4xyz}{2x^2y + \cos z}, \quad \frac{\partial z}{\partial y} = -\frac{\Phi_y}{\Phi_z} = -\frac{2x^2z}{2x^2y + \cos z}. \quad (2.132)$$

Notice that the derivatives are still functions of z and therefore if we want to obtain their value at a particular point (x, y) we need to solve first (2.130) and then substitute the corresponding value of z in (2.132). For example we found in the previous example (see previous page) that $z = f(1, 1) = 0$, therefore the partial derivatives at the point $(1, 1)$ are

$$\left. \frac{\partial z}{\partial x} \right|_{(1,1)} = -\frac{\Phi_x(1, 1)}{\Phi_z(1, 1)} = 1, \quad \left. \frac{\partial z}{\partial y} \right|_{(1,1)} = -\frac{\Phi_y(1, 1)}{\Phi_z(1, 1)} = 0. \quad (2.133)$$

Example 2: Consider now the same function of example 1 and assume in addition that the variables x and y are functions of two other variables u and v as

$$x = \frac{u^2 - v^2}{2} \quad \text{and} \quad y = uv. \quad (2.134)$$

Obtain the partial derivatives $\partial z/\partial u$ and $\partial z/\partial v$.

In this example we have to combine the results (2.128)-(2.129) with the chain rules we learnt in previous sections. The chain rule tells us that

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}, \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}, \end{aligned} \quad (2.135)$$

and the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ were computed in (2.132). Therefore, we only need to compute

$$\frac{\partial x}{\partial u} = u, \quad \frac{\partial x}{\partial v} = -v, \quad \frac{\partial y}{\partial u} = v \quad \text{and} \quad \frac{\partial y}{\partial v} = u. \quad (2.136)$$

Plugging (2.132) and (2.136) in (2.135) we obtain

$$\frac{\partial z}{\partial u} = -\frac{3x^2 + 4xyz}{2x^2y + \cos z}u - \frac{2x^2z}{2x^2y + \cos z}v, \quad (2.137)$$

$$\frac{\partial z}{\partial v} = \frac{2x^2 + 4xyz}{2x^2y + \cos z}v - \frac{2x^2z}{2x^2y + \cos z}u, \quad (2.138)$$

and if we replace x and y in terms of u and v we obtain,

$$\frac{\partial z}{\partial u} = \frac{(u^2 - v^2)/2}{(u^2 - v^2)^2 uv + 2 \cos z} [-(3(u^2 - v^2) + 8uvz)u - (u^2 - v^2)zv], \quad (2.139)$$

$$\frac{\partial z}{\partial v} = \frac{u^2 - v^2}{(u^2 - v^2)^2 uv + 2 \cos z} [(u^2 - v^2 + 4uvz)v - (u^2 - v^2)zu]. \quad (2.140)$$