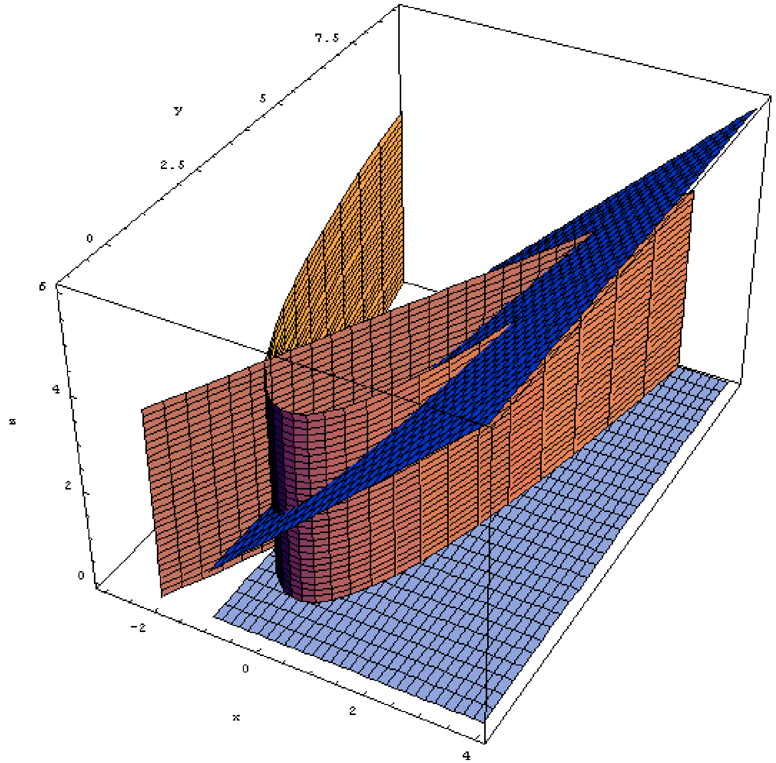
CITY UNIVERSITY
LONDON

REVISION CALCULUS 2009/10



FUNCTIONS OF TWO VARIABLES

$$f : \mathcal{D}(f) \in \mathbb{R}^2 \quad \longrightarrow \quad \mathcal{R}(f) \in \mathbb{R}$$
$$(x, y) \in \mathcal{D}(f) \quad \longrightarrow \quad f(x, y) \in \mathcal{R}(f)$$

- Domain of f: set of points (x, y) at which the function is well defined, e. g.

$$f(x, y) = \frac{1}{x - y} \quad \Rightarrow \quad \mathcal{D}(f) = \{(x, y) : x \neq y\}$$

- Range of f: range of values the function takes in its domain, e. g.

$$f(x, y) = \sqrt{9 - x^2 - y^2} \quad \Rightarrow \quad \mathcal{R}(f) = \{0 \leq f(x, y) \leq 3\}$$
$$f(x, y) = \sqrt{9 - x^2 - y^2} \quad \Rightarrow \quad \mathcal{D}(f) = \{(x, y) : x^2 + y^2 \leq 9\}$$

- Limits: The limit

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

exists only if it is unique. For example, the limit

$$\lim_{(x,y)\to(1,2)} \frac{2(x-1)(y-2)}{(x-1)^2+(y-2)^2}$$

$$= \lim_{(x,k(x-1)+2)\to(1,2)} \frac{2k(x-1)^2}{(x-1)^2+k^2(x-1)^2} = \frac{2k}{1+k^2},$$

does not exist.

PARTIAL DERIVATIVES

They can be computed in two ways:

- Definition through a limit:

$$f_x = \frac{\partial f}{\partial x} \quad = \quad \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h},$$
$$f_y = \frac{\partial f}{\partial y} \quad = \quad \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

- By using our knowledge of the derivatives of elementary functions, e.g. for

$$f(x,y) = x^2 \cos y + 2xy,$$

  the 1st order partial derivatives are

$$f_x = \frac{\partial f}{\partial x} = \cos y \frac{dx^2}{dx} + 2y \frac{dx}{dx} = 2x \cos y + 2y,$$
$$f_y = \frac{\partial f}{\partial y} = x^2 \frac{d \cos y}{dy} + 2x \frac{dy}{dy} = -x^2 \sin y + 2x.$$

CHAIN RULES

Given a function

$$f(x(t), y(t)),$$

the chain rule tells us that the partial derivative f_t can be obtained as

$$f_t = \frac{\partial f}{\partial t} \quad = \quad \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \equiv f_x x_t + f_y y_t,$$

Given a function

$$f(x(s,t), y(s,t)),$$

the chain rule tells us that the partial derivatives f_s and f_t can be obtained as

$$f_s = \frac{\partial f}{\partial s} \quad = \quad \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \equiv f_x x_s + f_y y_s,$$

$$f_t = \frac{\partial f}{\partial t} \quad = \quad \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \equiv f_x x_t + f_y y_t,$$

provided that f_x and f_y are continuous functions.

Consequently:

$$\begin{aligned}
\frac{\partial f_x}{\partial s} &= \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial s} \equiv f_{xx}x_s + f_{yx}y_s, \\
\frac{\partial f_x}{\partial t} &= \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial t} \equiv f_{xx}x_t + f_{yx}y_t, \\
\frac{\partial f_y}{\partial s} &= \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial s} \equiv f_{xy}x_s + f_{yy}y_s, \\
\frac{\partial f_y}{\partial t} &= \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial t} \equiv f_{xy}x_t + f_{yy}y_t.
\end{aligned}$$

These formulae will be important if we compute for example

$$f_{ss} = \frac{\partial(f_x x_s + f_y y_s)}{\partial s} = \frac{\partial f_x}{\partial s}x_s + f_x\frac{\partial x_s}{\partial s} + \frac{\partial f_y}{\partial s}y_s + f_y\frac{\partial y_s}{\partial s},$$

Example:

$$x = \sin(st) \qquad y = \cos(st) \quad \Rightarrow$$

$$x_s = t\cos(st), \quad x_t = s\cos(st), \quad y_s = -t\sin(st) \quad y_t = -s\sin(st).$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}(t\cos(st)) + \frac{\partial f}{\partial y}(-t\sin(st)),$$
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}(s\cos(st)) + \frac{\partial f}{\partial y}(-s\sin(st))$$

$$\frac{\partial^2 f}{\partial s^2} = \frac{\partial}{\partial s}\left(f_x(t\cos(st)) + f_y(-t\sin(st))\right)$$
$$= \frac{\partial f_x}{\partial s}(t\cos(st)) + f_x\frac{\partial(t\cos(st))}{\partial s} + \frac{\partial f_y}{\partial s}(-t\sin(st))$$
$$+f_y\frac{\partial(-t\sin(st))}{\partial s},$$

with

$$
\begin{aligned}
\frac{\partial f_x}{\partial s} &= \frac{\partial f_x}{\partial x}(t\cos(st)) + \frac{\partial f_x}{\partial y}(-t\sin(st)) \\
&= f_{xx}(t\cos(st)) + f_{yx}(-t\sin(st)), \\
\frac{\partial (t\cos(st))}{\partial s} &= -t^2\sin(st), \\
\frac{\partial f_y}{\partial s} &= \frac{\partial f_y}{\partial x}(t\cos(st)) + \frac{\partial f_y}{\partial y}(-t\sin(st)) \\
&= f_{xy}(t\cos(st)) + f_{yy}(-t\sin(st)), \\
\frac{\partial (-t\sin(st))}{\partial s} &= -t^2\cos(st).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial^2 f}{\partial s^2} &= -f_x(t^2\sin(st)) - f_y(t^2\cos(st)) + t^2\cos^2(st)f_{xx} \\
&+ t^2\sin^2(st)f_{yy} - t^2\sin(2st)f_{xy},
\end{aligned}
$$

APPROXIMATIONS OF FUNCTIONS ABOUT A POINT

- First differential:

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy = f_x dx + f_y dy,$$

The value of the function at a point (x, y) near the point (x_0, y_0) would be approximately:

$$f(x, y) \approx f(x_0, y_0) + df = f(x_0, y_0) + f_x(x_0, y_0)dx + f_y(x_0, y_0)dy,$$

where

$$dx = x - x_0, \qquad dy = y - y_0.$$

For example, if

$$f(x,y) = x^2y^4,$$

and we want to obtain the value of the function at $f(1.05, 1.06)$ then we can use

$$f(1.05, 1.06) \approx f(1,1) + df = f(1,1) + (0.05)f_x(1,1) + (0.06)f_y(1,1).$$

Since

$$f_x = 2xy^4, \qquad f_y = 4x^2y^3 \quad \Rightarrow \quad f_x(1,1) = 2, \quad f_y(1,1) = 4,$$

we obtain

$$f(1.05, 1.06) \quad \approx \quad 1 + 2(0.05) + 4(0.06) = 1 + 0.1 + 0.24 = 1.34.$$

The exact value of the function at this point is

$$f(1.05, 1.06) = (1.05)^2(1.06)^4 = 1.39188,$$

so the approximation is quite good!

- Taylor expansion about the point (x_0, y_0) (up to second-order terms):

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ f_{yx}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2 \\ &+ \frac{1}{2} f_{xx}(y_0,x_0)(x-x_0)^2 + \cdots \end{aligned}$$

For example, we consider the function

$$f(x,y) = \sqrt{x^2+y^3},$$

and we expand it about the point $(1,2)$ until second-order terms. We need:

$$\begin{aligned} f(1,2) &= 3, \\ f_x &= \frac{2x}{2\sqrt{x^2+y^3}} \quad \Rightarrow \quad f_x(1,2) = \frac{1}{3}, \\ f_y &= \frac{3y^2}{2\sqrt{x^2+y^3}} \quad \Rightarrow \quad f_y(1,2) = 2, \end{aligned}$$

$$
\begin{aligned}
f_{xx} &= \frac{y^3}{(x^2+y^3)^{3/2}} \quad \Rightarrow \quad f_{xx}(1,2) = \frac{8}{27} \\
f_{yy} &= \frac{3y(4x^2+y^3)}{4(x^2+y^3)^{3/2}} \quad \Rightarrow \quad f_{yy}(1,2) = \frac{2}{3}, \\
f_{xy} &= f_{yx} = -\frac{3xy^2}{2(x^2+y^3)^{3/2}} \quad \Rightarrow \quad f_{xy}(1,2) = -\frac{2}{9}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(x,y) &= 3 + \frac{1}{3}(x-1) + 2(y-2) - \frac{2}{9}(x-1)(y-2) + \frac{1}{3}(y-2)^2 \\
&+ \frac{4}{27}(x-1)^2 = \frac{1}{27}(-8 + 4x^2 + 13x - 6xy + 24y + 9y^2).
\end{aligned}
$$

DERIVATIVES OF IMPLICIT FUNCTIONS

Functions are defined implicitly when they are given by means of a constrain,

$$\Phi(x, y, z) = 0,$$

which involves the independent variables x, y and the function $z = f(x, y)$.
Employing

$$d\Phi = \Phi_x dx + \Phi_y dy + \Phi_z dz = 0,$$

$$dz = f_x dx + f_y dy,$$

one can obtain the following formulae:

$$\frac{\partial z}{\partial x} = -\frac{\Phi_x}{\Phi_z},$$

$$\frac{\partial z}{\partial y} = -\frac{\Phi_y}{\Phi_z}.$$

Example: Given

$$\Phi(x,y,z) = (x^2+y^2+z^2)\sin(z+x+y) = 0,$$

we have

$$\begin{aligned}
\Phi_x &= 2x\sin(x+y+z) + (x^2+y^2+z^2)\cos(z+x+y),\\
\Phi_y &= 2y\sin(x+y+z) + (x^2+y^2+z^2)\cos(z+x+y),\\
\Phi_z &= 2z\sin(x+y+z) + (x^2+y^2+z^2)\cos(z+x+y).
\end{aligned}$$

Therefore

$$\frac{\partial z}{\partial x} = -\frac{\Phi_x}{\Phi_z} = -\frac{2x\sin(x+y+z)+(x^2+y^2+z^2)\cos(z+x+y)}{2z\sin(x+y+z)+(x^2+y^2+z^2)\cos(z+x+y)},$$

$$\frac{\partial z}{\partial y} = -\frac{\Phi_y}{\Phi_z} = -\frac{2y\sin(x+y+z)+(x^2+y^2+z^2)\cos(z+x+y)}{2z\sin(x+y+z)+(x^2+y^2+z^2)\cos(z+x+y)}.$$

CLASSIFICATION OF STATIONARY POINTS

$AC - B^2$	> 0	< 0	0
$A > 0$	minimum	saddle point	inconclusive
$A < 0$	maximum	saddle point	inconclusive
$A = 0$		saddle point	inconclusive

with

$$\begin{aligned} A &= f_{xx}(x_0, y_0), \qquad C = f_{yy}(x_0, y_0), \qquad B = f_{xy}(x_0, y_0) \\ f_x(x_0, y_0) &= f_y(x_0, y_0) = 0. \end{aligned}$$

Example: Let

$$f(x, y) = x^2 + y^2 \quad \Rightarrow$$

$$f_x = 2x, \qquad f_y = 2y, \qquad f_{xx} = 2, \qquad f_{yy} = 2, \qquad f_{xy} = 0.$$

Therefore, the only point we have to consider is $(x, y) = (0, 0)$ where

$$f_x(0,0) = f_y(0,0) = 0.$$

At this point

$$AC - B^2 = (2)(2) - 0 = 4 > 0$$

and

$$A = 2 > 0$$

$(0, 0)$ is a minimum and the only stationary point of this function.

LAGRANGE MULTIPLIERS

The method of Lagrange multipliers allows us to treat “constrained extreme-value problems”. This means finding the minimum or maximum value of a function

$$f(x, y, z),$$

subject to a certain constraint

$$\phi(x, y, z) = 0.$$

The problem can be solved by constructing a linear combination of these two functions

$$f + \lambda\phi,$$

where λ is called the Lagrange multiplier and imposing that

$$df + \lambda d\phi = 0 \quad \Rightarrow \quad f_x + \lambda\phi_x = f_y + \lambda\phi_y = f_z + \lambda\phi_z = 0.$$

These 3 equations, together with the constraint itself, give a system of 4 equations which can be solved for the unknowns x, y, z, λ.

Example: Obtain the maximum value of the function

$$f(x, y, z) = xy + z^3,$$

if x, y, z satisfy

$$x + y + z = 1 \quad \Rightarrow \quad \phi(x, y, z) = 0 = x + y + z - 1.$$

$$\begin{aligned} f_x &= y, \qquad f_y = x, \qquad f_z = 3z^2, \\ \phi_x &= 1, \qquad \phi_y = 1, \qquad \phi_z = 1. \end{aligned}$$

And so we have to solve the equations

$$\begin{aligned} y + \lambda &= 0, \\ x + \lambda &= 0, \\ 3z^2 + \lambda &= 0, \\ x + y + z - 1 &= 0. \end{aligned}$$

From the 3 first equations we get

$$\lambda = -y = -x = -3z^2 \qquad \Rightarrow \qquad x = y = 3z^2,$$

substituting this into the last equation we obtain

$$z + 6z^2 = 1 \qquad \Rightarrow \qquad z = -\frac{1}{2}, \frac{1}{3}.$$

Therefore the solutions are

$$(x, y, z) = (3/4, 3/4, -1/2) \qquad \lambda = -3/4,$$

and

$$(x, y, z) = (1/3, 1/3, 1/3) \qquad \lambda = -1/3,$$

since

$$f(3/4, 3/4, -1/2) = 9/16 - 1/8 = 7/16 \approx 0.42,$$

$$f(1/3, 1/3, 1/3) = 1/9 - 1/27 = 2/27 \approx 0.07$$

the point $(3/4, 3/4, -1/2)$ is the solution to the problem!

INTEGRATION OF FUNCTIONS OF SEVERAL VARIABLES

- 2-dimensional integrals:

$$\int\int_R f(x,y)dxdy$$

We saw basically two types of integrals, depending on the integration region R:

$$R = \{(x,y):\ a \leq x \leq b \quad c \leq y \leq d\}$$

and

$$R = \{(x,y):\ a \leq x \leq b \quad g_1(x) \leq y \leq g_2(x)\}$$

or

$$R = \{(x,y):\ a \leq y \leq b \quad h_1(y) \leq x \leq h_2(y)\}$$

Example:

$$\int_{x=0}^{x=3} dx \int_{y=0}^{y=2} xy^2 dy.$$

If we do the y-integral first:

$$\int_{y=0}^{y=2} y^2 dy = \left[\frac{y^3}{3}\right]_{y=0}^{y=2} = \frac{8}{3},$$

and then the integral in x

$$\int_{x=0}^{x=3} x dx = \left[\frac{x^2}{2}\right]_{x=0}^{x=3} = \frac{9}{2}.$$

Therefore the value of the integral would be

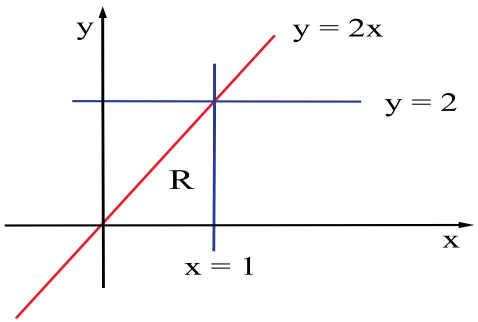
$$\int_{x=0}^{x=3} dx \int_{y=0}^{y=2} xy^2 dy = \left(\frac{8}{3}\right)\left(\frac{9}{2}\right) = 12.$$

The second type of integrals is a bit more complicated because the order of integration matters. Therefore if we want to change the order of integration, we also have to change our description of the integration region R.

Example: Sketch the region of integration in the double integral

$$I = \int_{y=0}^{y=2} dy \int_{x=y/2}^{x=1} \cos(x^2) dx.$$

By changing the order of integration, evaluate I.



From the picture it is easy to see that changing the order of integration we obtain

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=2x} \cos(x^2) dy.$$

The integral in y gives

$$\int_{y=0}^{y=2x} \cos(x^2) dy = \left[y \cos(x^2) \right]_0^{2x} = 2x \cos(x^2).$$

Plugging this result into the second integral we obtain

$$I = \int_{x=0}^{x=1} 2x \cos(x^2) dx = \left[\sin(x^2) \right]_0^1 = \sin(1).$$

CHANGES OF VARIABLES AND JACOBIANS

We have seen 3 changes of coordinates which are especially important:

- the **Polar coordinate system**:

$$x = r\cos\theta \qquad\qquad y = r\sin\theta,$$

the element of surface is: $dxdy = rdrd\theta$

- the **Cylindrical coordinates**:

$$x = r\cos\theta \qquad y = r\sin\theta, \qquad z = z$$

the element of volume is: $dxdydz = rdrd\theta dz$

- the **Spherical coordinates**:

$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi.$$

the element of volume is: $dxdydz = r^2\sin\phi drd\theta d\phi$

Elements of surface and volume are obtained from the **Jacobian determinant**.

The Jacobian determinant is built out of the derivatives of the old variables with respect to the new ones:

$$(x, y) \to (u, v)$$

then

$$dxdy = |J|dudv, \qquad J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

and if we have 3 variables,

$$(x, y, z) \to (u, v, t)$$

then

$$dxdydz = |J|dudvdt$$

with

$$J = \begin{vmatrix} x_u & x_v & x_t \\ y_u & y_v & y_t \\ z_u & z_v & z_t \end{vmatrix}$$

Example: Consider the change of variables:

$$x = s + 3t, \qquad y = s - 3t,$$

then the Jacobian would be

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = -3 - 3 = -6,$$

Therefore the new element of surface would be

$$dxdy = 6dsdt$$

<u>3D INTEGRALS</u>

These are harder to do because it is more difficult to visualize the integration region! However we have to try to get at least an idea of how this region looks like:

- Sketch integration region (as good as we can)
- Change coordinates when appropriate (usually this will be said)
- Be familiar with quadratic surfaces such as spheres $x^2 + y^2 + z^2 = a^2$, cones $z^2 = x^2 + y^2$, cylinders $x^2 + y^2 = 1$ and paraboloids $z = x^2 + y^2$!

Example: Show that the equation of the semicircle $x^2 + y^2 - ay = 0$ with $x \geq 0$ in the polar coordinates takes the form

$$r = a \sin \theta \qquad\qquad 0 \leq \theta \leq \frac{\pi}{2}.$$

Hence using the cylindrical coordinates find the volume of the solid that is inside of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1,$$

above the xy-plane and inside the vertical cylinder $x^2 + y^2 - ay = 0, x \geq 0$.

(a) In polar coordinates $x = r\cos\theta, y = r\sin\theta$ the equation of the semi-circle becomes

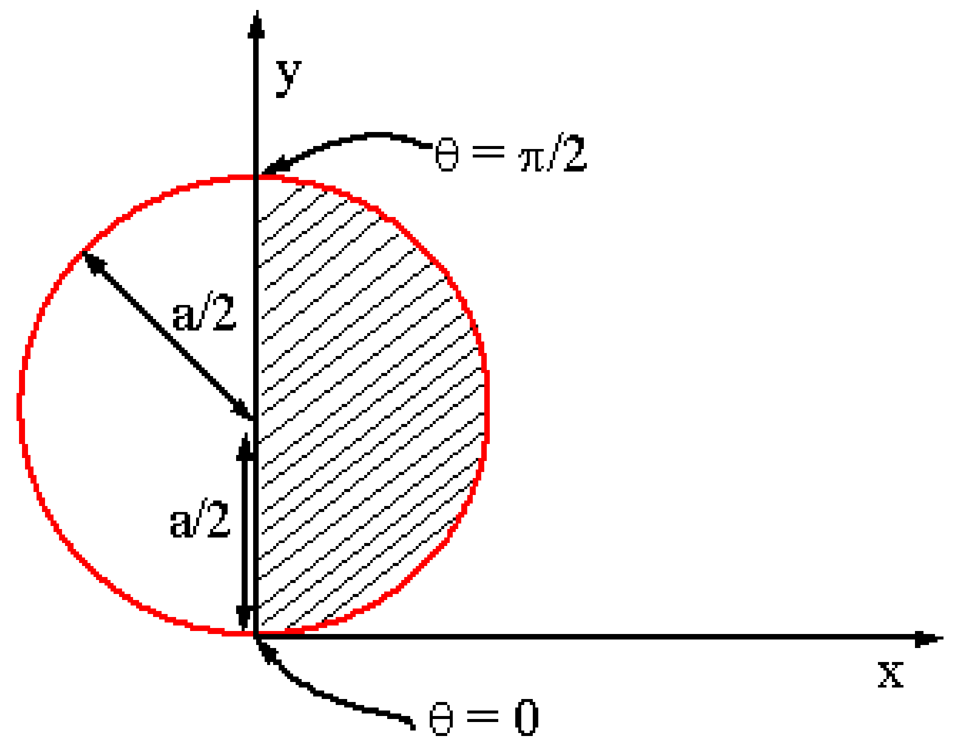
$$x^2 + y^2 - ay = 0 \quad \Leftrightarrow \quad r^2 - ar\sin\theta = 0 \quad \Leftrightarrow \quad r = a\sin\theta.$$

We notice that

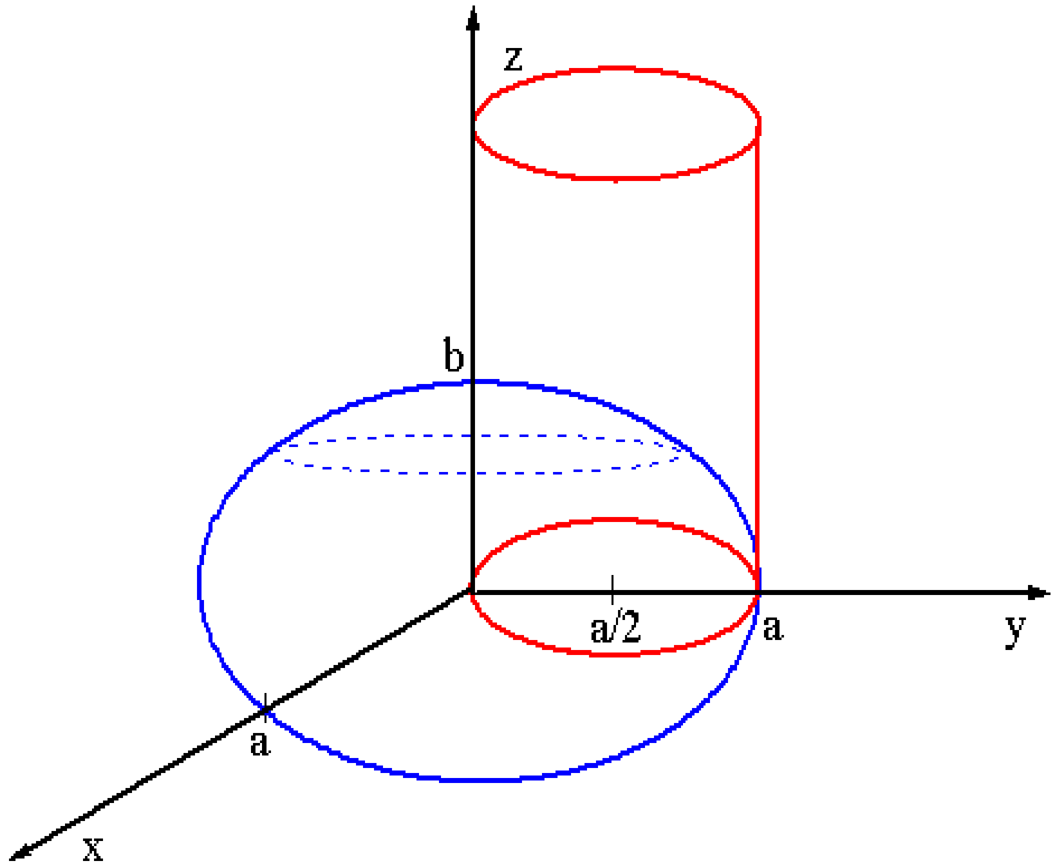
$$x^2 + y^2 - ay = 0 \quad \Leftrightarrow \quad x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4},$$

this

is a circle centered at the point $(0, a/2)$ of radius $a/2$. For $x \geq 0$ it is a semicircle!



y
θ = π/2
a/2
a/2
x
θ = 0


z
b
a/2
a
y
a
x

$$R = \left\{ (r, \theta, z) \,|\, 0 \le r \le a \sin\theta, \quad 0 \le \theta \le \pi/2, \quad 0 \le z \le b\sqrt{1 - \frac{r^2}{a^2}} \right\},$$

The element of volume in cylindrical coordinates is

$$dx\,dy\,dz = r\,dr\,d\theta\,dz,$$

and therefore the volume is

$$V = \int_{\theta=0}^{\theta=\pi/2} d\theta \int_{r=0}^{r=a\sin\theta} r\,dr \int_{z=0}^{z=b\sqrt{1-r^2/a^2}} dz.$$

The integral in z gives

$$\int_{z=0}^{z=b\sqrt{1-r^2/a^2}} dz = b\sqrt{1-r^2/a^2}.$$

Plugging this into the r-integral we have

$$\int_{r=0}^{r=a\sin\theta} rb\sqrt{1-r^2/a^2}dr,$$

changing variables to $t = 1 - r^2/a^2$ we obtain

$$dt = -2r/a^2 dr.$$

The integration limit $r = 0$ corresponds to $t = 1$ and $r = a \sin\theta$ corresponds to $t = 1 - \sin^2\theta = \cos^2\theta$, so that the integral becomes

$$-\frac{a^2 b}{2}\int_{t=1}^{t=\cos^2\theta} t^{1/2}dt = -\frac{a^2 b}{3}\left[t^{3/2}\right]_{t=1}^{t=\cos^2\theta} = -\frac{a^2 b}{3}(\cos^3\theta - 1).$$

Substituting this result into the θ-integral we obtain

$$V = \frac{a^2 b}{3}\int_{\theta=0}^{\theta=\pi/2}(1-\cos^3\theta)d\theta = \frac{a^2 b}{18}\left(3\pi - 4\right).$$

METHOD OF VARIATION OF PARAMETERS

We have seen a new method to solve second-order linear differential equations of the type:

$$y'' + ay' + by = R(x).$$

The general solution of such an equation is given by

$$y = y_h + y_i^p,$$

where

$$y_h(x) = c_1 u_1(x) + c_2 u_2(x),$$

is the general solution of the homogeneous equation

$$y'' + ay' + by = 0,$$

and y_i^p is a particular solution of the inhomogeneous equation. This solution can be found by using the method of variation of parameters

$$y_p^i(x) = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \quad \text{and} \quad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx,$$

and

$$W(x) = u_1(x)u_2'(x) - u_2(x)u_1'(x).$$

Example: Suppose we want to solve

$$y'' + 2y' + 2y = \frac{e^{-x}}{\cos^3(x)}.$$

Homogeneous equation: we try solutions $y = ce^{mx}$,

$$m^2 + 2m + 2 = 0 \quad \Rightarrow \quad m = -1 \pm i,$$

that is

$$y_h = c_1 e^{-x} \sin x + c_2 e^{-x} \cos x,$$

therefore

$$u_1(x) = e^{-x} \sin x \qquad u_2(x) = e^{-x} \cos x,$$

and

$$u_1'(x) = -e^{-x} \sin x + e^{-x} \cos x \qquad u_2'(x) = -e^{-x} \cos x - e^{-x} \sin x.$$

From these functions we can prove that

$$W(x) = -e^{-2x}.$$

With this information we can now find the particular solution of the inhomogeneous equation:

$$y_p^i(x) = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = \int \frac{1}{\cos^2(x)} dx \quad \text{and} \quad v_2(x) = -\int \frac{\sin x}{\cos^3 x} dx.$$

To do the second integral we can change variables as $t = \cos x$

$$v_2(x) = -\int t^{-3} dt = \frac{1}{2} t^{-2} = \frac{1}{2\cos^2 x},$$

and the first integral is direct

$$v_1(x) = \tan x.$$

$$y = e^{-x} \sin x (c_1 + \tan x) + e^{-x} \cos x (c_2 + 1/(2\cos^2 x))$$