Sheet 3: chain rules

1. If $z = f(x,y)$ and $x = 2s + 3t$ and $y = 3s - 2t$ find
$$\frac{\partial^2 z}{\partial s^2}, \qquad \frac{\partial^2 z}{\partial s \partial t} \quad \text{and} \quad \frac{\partial^2 z}{\partial t^2}.$$

2. If $x = e^s \cos t$, $y = e^s \sin t$ and $z = f(x,y)$, show that
$$\frac{\partial^2 f}{\partial s^2} + \frac{\partial^2 f}{\partial t^2} = (x^2 + y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

3. The atmospheric temperature depends on the position and time. If we denote the position by three spatial coordinates x, y, z (measured in Km) and time by t (measured in hours), then the temperature T is a function of four variables $T(x,y,z,t)$.

   (a) If a thermometer is attached to a weather balloon that moves through the atmosphere on a path with parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$, what is the rate of change at time t of the temperature measured by the thermometer?

   (b) Find the rate of change of the measured temperature at time $t = 1$ if
$$T(x,y,z,t) := \frac{100}{5 + x^2 + y^2}\left(1 + \sin\frac{\pi t}{12}\right) - 20(1 + z^2),$$
   in degrees Celsius, and if the balloon moves along the curve
$$x = f(t) = t, \qquad y = g(t) = 2t \quad \text{and} \quad z = h(t) = t - \frac{t^4}{2}.$$

4. **Laplace's equation in polar coordinates.** Consider a function of two variables $f(x,y)$ which has continuous partial derivatives of 1st and 2nd order. Suppose that x and y can be expressed in terms of some new coordinates r and θ as follows
$$x = x(r,\theta) = r\cos(\theta) \quad \text{and} \quad y = y(r,\theta) = r\sin(\theta).$$
   The new coordinates (r,θ) are called **polar coordinates** (see figure 7). By employing the chain rule prove that the following identity holds
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

   Note: The operation
$$\triangle f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$
   is called the **Laplacian** of f.

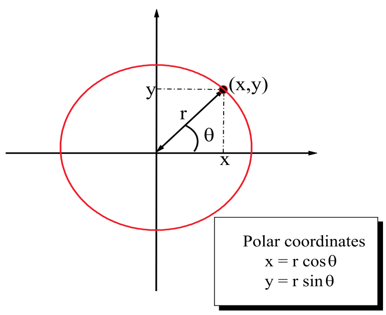

Figure 7: Relation between the **polar coordinates** (r,θ) and the **cartesian coordinates** (x,y).