Sheet 4: differentials, implicit functions and Taylor expansion

1. Calculate the value of the total differential du of the function of three variables
$$u(x,y,z) = \frac{xy}{\sqrt{x^2+y^2+z^2}},$$
at the point $(x,y,z) = (1,3,-2)$ and for $\Delta x = 1/2$, $\Delta y = 1/4$ and $\Delta z = -1/4$.

2. Given the function
$$g(x,y) = \frac{1}{x^2+y^2-1},$$
Write down an expression for the first order differential dg. Use the differential to estimate the value of the function at the point $(1.01, -1.003)$.

3. Given the functions
$$(a)\ f(x,y) = e^{x-y} - e^{y-x}, \qquad (b)\ g(x,y) = x^y y^x.$$
Write down a formula for their second order differential and compute its value for $(x,y) = (1,2)$ and $(dx,dy) = (0.01, -0.01)$.

4. Compute $\partial z/\partial x$ and $\partial z/\partial y$ if
$$z^2 - 2xy^3 = \frac{xz}{y}.$$
Obtain the value of $z > 0$ at the point $(x,y) = (1,1)$ and the numerical value of $\partial z/\partial x$ and $\partial z/\partial y$ at this point.

5. The constraint
$$\Phi(x,y,z) = x^2+y^2+z^2-4 = 0,$$
can be solved for $z = \sqrt{4-x^2-y^2}$ with $z > 0$. Obtain the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ in two possible ways: from the expression of z in terms of x and y and from the formulae for the derivatives of implicit functions that we have seen in the lecture. Show that both results are the same.

6. Find a 2nd order polynomial approximation to the function
$$f(x,y) = \sqrt{x^2+y^3},$$
near the point $(1,2)$. Hence use this approximation to estimate the value of $\sqrt{(1.02)^2+(1.97)^3}$.

7. Obtain the Taylor expansion up to 3th order terms of the function
$$f(x,y) = \frac{1}{2+x-2y},$$
near the point $(2,1)$.

8. Obtain the Taylor expansion up to 2nd order terms of the function
$$f(x,y) = \ln(x^2+y^2),$$
near the point $(1,0)$.