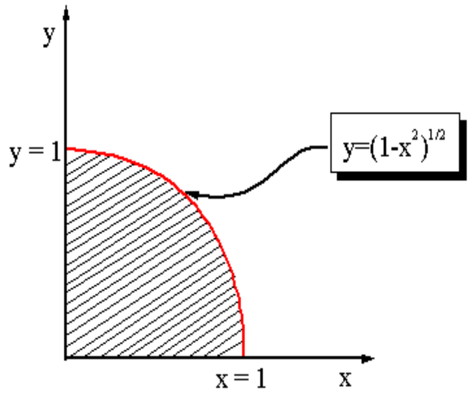
CALCULUS 1999: EXAM SOLUTIONS

1. (a) The integration region is the dashed area below. Notice that the curve $y = \sqrt{1-x^2}$ can als(



In polar coordinates we have

$$x = r\cos\theta \qquad y = r\sin\theta,$$

and the element of surface is

$$dxdy = rdrd\theta.$$

Looking at the integration region it is easy to see that in polar coordinates it corresponds to

$$R := \{(r,\theta) \mid 0 \le \theta \le \pi/2, \quad 0 \le r \le 1\}.$$

Finally, in polar coordinates, the function inside the integral becomes

$$e^{-(x^2+y^2)} = e^{-r^2}.$$

So the integral is

$$I = \int_{\theta=0}^{\theta=\pi/2} d\theta \int_{r=0}^{r=1} re^{-r^2} dr.$$

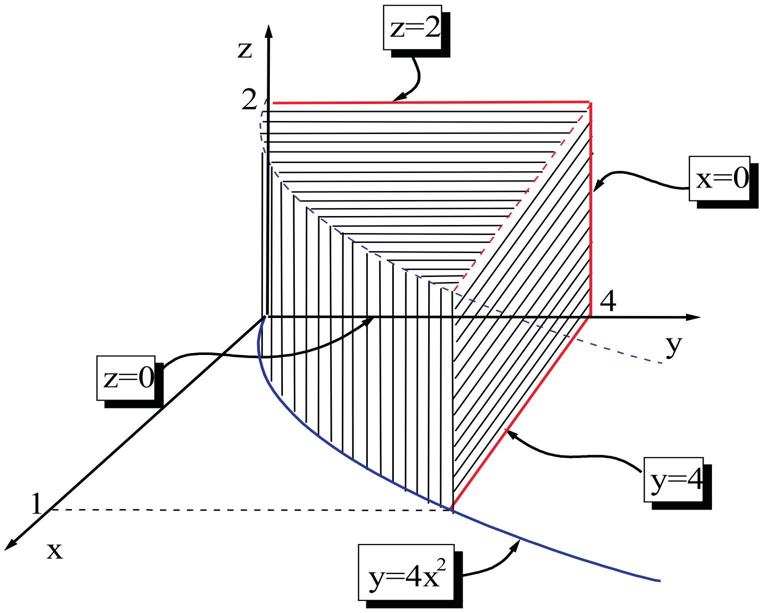
The integral in r is simply

$$\int_{r=0}^{r=1} re^{-r^2} dr = -\frac{1}{2}\left[e^{-r^2}\right]_0^1 = \frac{1-e^{-1}}{2} = \frac{e-1}{2e}.$$

Therefore

$$I = \frac{e-1}{2e}\int_{\theta=0}^{\theta=\pi/2} d\theta = \frac{e-1}{2e}\left[\theta\right]_0^{\pi/2} = \frac{\pi(e-1)}{4e}.$$

(b) As usual we start by sketching the integration region



From the picture and the information given by the problem we can deduce that the integration region is

$$R = \{(x, y, z) : 0 \le x \le 1, \quad 4x^2 \le y \le 4, \quad 0 \le z \le 2.\}$$

The integral is

$$I = 2\int_{x=0}^{x=1} x dx \int_{y=4x^2}^{4} dy \int_{z=0}^{2} dz.$$

The integral in z is

$$\int_{z=0}^{2} dz = [z]_0^2 = 2.$$

The integral in y is

$$\int_{y=4x^2}^{4} dy = [y]_{4x^2}^{4} = 4(1 - x^2).$$

Therefore, the final result is

$$I = 16 \int_{x=0}^{x=1} x(1 - x^2) dx = 16 \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 16\left(\frac{1}{2} - \frac{1}{4}\right) = 4.$$

2. (a) For this function the first order partial derivatives are

$$\begin{aligned} f_x &= 3x^2 + y^2 - 24x + 21, \\ f_y &= 2xy - 4y. \end{aligned}$$

The first thing we have to do is finding the points at which these derivatives vanish

$$f_y = 0 \quad \Rightarrow \quad y(x - 2) = 0 \quad \Rightarrow y = 0 \quad \text{or} \quad x = 2.$$

For $y = 0$ (which is one of the solutions of the previous equation) f_x will vanish if

$$f_x(x, y = 0) = 0 = 3x^2 - 24x + 21 = 0 \;\Rightarrow\; x = \frac{24 \pm 18}{6} = 7, 1,$$

and for $x = 2$ (which is the other solution of $f_y = 0$) we would obtain

$$f_x(x = 2, y) = 0 = 12 + y^2 - 48 + 21 \quad \Rightarrow \quad y = \pm\sqrt{15}.$$

Therefore, putting all these solutions together we have the following 4 points:

$$(x, y) = (1, 0),\; (7, 0),\; (2, \sqrt{15}) \quad \text{and} \quad (2, -\sqrt{15}).$$

The next step is to compute the second order partial derivatives

$$\begin{aligned} A &= f_{xx} = 6x - 24, \\ B &= f_{xy} = f_{yx} = 2y, \\ C &= f_{yy} = 2x - 4, \end{aligned}$$

therefore

$$AC - B^2 = (6x - 24)(2x - 4) - 4y^2,$$

and we have to study the sign of this quantity in order to classify the stationary points of the function:

The point $(1, 0)$: At this point

$$\begin{aligned} AC - B^2 &= (6 - 24)(2 - 4) = 36 > 0, \\ A &= 6 - 24 = -18 < 0, \end{aligned}$$

therefore this point is a **maximum**.

The point $(7, 0)$: At this point

$$\begin{aligned} AC - B^2 &= (42 - 24)(14 - 4) = 180 > 0, \\ A &= 42 - 24 = 18 > 0, \end{aligned}$$

therefore this point is a **minimum**.

The point $(2, \sqrt{15})$: At this point

$$AC - B^2 = (12 - 24)(4 - 4) - 60 = -60 < 0,$$

therefore this point is a **saddle point**.

The point $(2, -\sqrt{15})$: At this point

$$AC - B^2 = (12 - 24)(4 - 4) - 60 = -60 < 0,$$

therefore this point is also a **saddle point**.

(b) The Taylor expansion of a function of two variables $f(x,y)$ around a point (x_0,y_0) up to second order terms is given by

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0,y_0) = (-1,-1)$ and

$$\begin{aligned} f_x &= (1+x+y)e^{x-y}, \quad f_y = (1-x-y)e^{x-y}, \quad f_{xx} = (2+x+y)e^{x-y}, \\ f_{yy} &= (-2+x+y)e^{x-y}, \qquad f_{xy} = f_{yx} = -(x+y)e^{x-y}. \end{aligned}$$

Therefore

$$\begin{aligned} f_x(-1,-1) &= -1, \quad f_y(-1,-1) = 3, \quad f_{xx}(-1,-1) = 0, \\ f_{yy}(-1,-1) &= -4, \qquad f_{xy}(-1,-1) = f_{yx}(-1,-1) = 2, \end{aligned}$$

and $f(-1,-1) = -2$. With this we obtain the following Taylor expansion

$$\begin{aligned} f(x,y) &= -2 - (x+1) + 3(y+1) - 2(y+1)^2 + 2(y+1)(x+1) \\ &= y + x - 2y^2 + 2xy. \end{aligned}$$

Therefore, the approximate value of $f(-0.9,-1.05)$ is

$$f(-0.9,-1.05) \simeq -0.9 - 1.05 - 2(1.05)^2 + 2(0.9)(1.05) = -2.265.$$

The Taylor expansion in terms of the displacements h and k is obtained simply by replacing $x = x_0 + h = h - 1$ and $y = y_0 + k = k - 1$ in our final formula. It gives

$$f(h,k) = k + h - 2 - 2(k-1)^2 + 2(k-1)(h-1).$$

3. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 + 1 = 0 \Rightarrow m = \pm i.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 \sin(x) + c_2 \cos(x),$$

therefore we identify

$$u_1(x) = \sin(x), \qquad u_2(x) = \cos(x).$$

For the second part of the problem we will need the Wronskian of these solutions which is

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -\sin^2(x) - \cos^2(x) = -1.$$

Therefore the Wronskian is indeed nowhere zero.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = x + \frac{1}{\cos x}, \qquad W(x) = -1,$$

therefore

$$v_1(x) = \int (x\cos x + 1)dx = x + \int x\cos x dx = x + x\sin x - \int \sin x dx = x + x\sin x + \cos x.$$

$$\begin{aligned} v_2(x) &= -\int (x\sin x + \tan x)dx = \ln|\cos x| - \int x\sin x dx = \ln|\cos x| + x\cos x - \int \cos x dx \\ &= \ln|\cos x| + x\cos x - \sin x, \end{aligned}$$

where for both integrals we have used integration by parts. Hence the general solution of the inhomogeneous equation is

$$\begin{aligned} y &= c_1\sin x + c_2\cos(x) + \sin x(x + x\sin x + \cos x) + \cos x(\ln|\cos x| + x\cos x - \sin x) \\ &= c_1\sin x + c_2\cos x + x(1 + \sin x) + \cos x\ln|\cos x|. \end{aligned}$$

with c_1, c_2 being arbitrary constants.

4. Here we need to use the chain rule for a function of two variables x, y which are changed to two new variables u, v. The relevant identities are

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \tag{0.1}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \tag{0.2}$$

For $x = (u + v)/2$ and $y = (u^2 + v^2)/4$ we have

$$\frac{\partial x}{\partial u} = \frac{\partial x}{\partial v} = \frac{1}{2}, \qquad \frac{\partial y}{\partial u} = \frac{u}{2}, \qquad \frac{\partial y}{\partial v} = \frac{v}{2}.$$

Plugging these derivatives into (0.1)-(0.2) we obtain

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{1}{2}f_x + \frac{u}{2}f_y, \\ \frac{\partial f}{\partial v} &= \frac{1}{2}f_x + \frac{v}{2}f_y. \end{aligned}$$

therefore

$$\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}=\frac{1}{2}f_x+\frac{u}{2}f_y+\frac{1}{2}f_x+\frac{v}{2}f_y=f_x+\frac{u+v}{2}f_y=f_x+xf_y,$$

as we wanted to prove. To prove the second identity we need to compute second order partial derivatives

$$\begin{aligned}\frac{\partial^2 f}{\partial u^2} &= \frac{1}{2}\frac{\partial}{\partial u}(f_x+uf_y)=\frac{1}{2}\frac{\partial f_x}{\partial u}+\frac{1}{2}f_y+\frac{u}{2}\frac{\partial f_y}{\partial u}\\ &= \frac{1}{4}(f_{xx}+uf_{yx})+\frac{1}{2}f_y+\frac{u}{4}(f_{xy}+uf_{yy}),\\ \frac{\partial^2 f}{\partial v^2} &= \frac{1}{2}\frac{\partial}{\partial v}(f_x+vf_y)=\frac{1}{2}\frac{\partial f_x}{\partial v}+\frac{1}{2}f_y+\frac{v}{2}\frac{\partial f_y}{\partial v}\\ &= \frac{1}{4}(f_{xx}+vf_{yx})+\frac{1}{2}f_y+\frac{v}{4}(f_{xy}+vf_{yy}).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2} &= \frac{1}{4}(f_{xx}+uf_{yx})+\frac{1}{2}f_y+\frac{u}{4}(f_{xy}+uf_{yy})+\frac{1}{4}(f_{xx}+vf_{yx})+\frac{1}{2}f_y+\frac{v}{4}(f_{xy}+vf_{yy})\\ &= \frac{1}{2}f_{xx}+\frac{u+v}{2}f_{xy}+\frac{u^2+v^2}{4}f_{yy}+f_y\\ &= \frac{1}{2}f_{xx}+xf_{xy}+yf_{yy}+f_y,\end{aligned}$$

as we wanted to prove.

(b) Let us consider an implicit function of two variables $z=f(x,y)$ and assume the existence of a constraint

$$F(x,y,z)=0,$$

which relates the function z to the two independent variables x and y. Since $F=0$ it is clear that also its total differential $dF=0$ must vanish. However the total differential is by definition

$$dF=\left(\frac{\partial F}{\partial x}\right)dx+\left(\frac{\partial F}{\partial y}\right)dy+\left(\frac{\partial F}{\partial z}\right)dz=0, \tag{0.3}$$

and in addition, z is a function of x and y, therefore its differential is given by

$$dz=\left(\frac{\partial z}{\partial x}\right)dx+\left(\frac{\partial z}{\partial y}\right)dy. \tag{0.4}$$

If we substitute (0.4) into (0.3) we obtain the equation

$$dF=0=\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right)dx+\left(\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\right)dy. \tag{0.5}$$

Since x and y are independent variables, equation (0.5) implies that each of the factors has to vanish separately, that is

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0.$$

Therefore we obtain,

$$\frac{\partial z}{\partial x} = -\frac{\left(\dfrac{\partial F}{\partial x}\right)}{\left(\dfrac{\partial F}{\partial z}\right)} = -\frac{F_x}{F_z},$$

$$\frac{\partial z}{\partial y} = -\frac{\left(\dfrac{\partial F}{\partial y}\right)}{\left(\dfrac{\partial F}{\partial z}\right)} = -\frac{F_y}{F_z}.$$

Employing now these formulae for the function $F(x, y, z) = \cos(x+y)+\sin(y+z) = 0$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{\sin(x+y)}{\cos(y+z)},$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-\sin(x+y)+\cos(y+z)}{\cos(y+z)} = \frac{\sin(x+y)}{\cos(y+z)} - 1.$$