CALCULUS 2000: EXAM SOLUTIONS

1. (a) The integration region is the triangle enclosed by the lines $y = \pi/2$, $x = 0$ and $x = y$. Changing the order of integration we obtain the integral

$$I = \int_{x=0}^{\pi/2} \sqrt{1 - k^2 \sin^2 x} dx \int_{y=x}^{\pi/2} \cos(2y) dy,$$

the integral in y gives

$$\int_{y=x}^{\pi/2} \cos(2y) dy = \frac{1}{2} \left[\sin(2y)\right]_x^{\pi/2} = \frac{1}{2}(0 - \sin(2x)) = -\frac{1}{2}\sin(2x).$$

Plugging this result into I we obtain

$$I = -\frac{1}{2} \int_{x=0}^{\pi/2} \sqrt{1 - k^2 \sin^2 x}\ \sin(2x) dx.$$

In order to do this integral it is convenient to introduce the change of variables

$$t = k \sin x \quad \Leftrightarrow \quad dt = k \cos x dx,$$

and use the identity $\sin(2x) = 2 \cos x \sin x$. We obtain then

$$I = -\frac{1}{k^2} \int_{t=0}^{t=k} t \sqrt{1 - t^2} dt = \frac{1}{k^2} \left[\frac{1}{3}(1 - t^2)^{3/2}\right]_0^k = \frac{(1 - k^2)^{3/2} - 1}{3k^2}.$$

(b) We start by computing the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -v/u^2 & 1/u \\ 0 & 1 \end{vmatrix} = -\frac{v}{u^2}.$$

Therefore

$$dx\, dy = |J| du\, dv = \frac{v}{u^2} du\, dv.$$

Now we have to transform the function we want to integrate,

$$\frac{y^2}{x^2} e^{y/x} = u^2 e^u,$$

and we have to find the new integration region

$$\begin{aligned} 0 \le x \le 1 \quad &\Leftrightarrow \quad 0 \le v \le u, \\ 0 \le y \le x \quad &\Leftrightarrow \quad 0 \le u \le 1. \end{aligned}$$

Therefore the integral we need to compute is

$$I = \int_{u=0}^{u=1} e^u du \int_{v=0}^{v=u} v\, dv.$$

The first integral is

$$\int_{v=0}^{v=u} v\,dv = \left[\frac{v^2}{2}\right]_{v=0}^{v=u} = \frac{u^2}{2},$$

and so

$$I = \frac{1}{2}\int_{u=0}^{u=1} u^2 e^u du.$$

This integral can be done by using integration by parts twice

$$\begin{aligned}
\int_{u=0}^{u=1} u^2 e^u du &= \left[u^2 e^u\right]_0^1 - \int_{u=0}^{u=1} 2ue^u du = e - \int_{u=0}^{u=1} 2ue^u du \\
&= e - \left[2ue^u\right]_0^1 + \int_{u=0}^{u=1} 2e^u du = e - 2e + \int_{u=0}^{u=1} 2e^u du \\
&= \left[2e^u\right]_0^1 - e = 2e - 2 - e = e - 2.
\end{aligned}$$

Therefore

$$I = \frac{e-2}{2}.$$

2. (a) The Taylor expansion of a function of two variables $f(x,y)$ around a point (x_0,y_0) up to second order terms is given by

$$\begin{aligned}
f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\
&+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0).
\end{aligned}$$

First we need to compute the 1st and 2nd order partial derivatives

$$\begin{aligned}
f_x &= 2e^{2x+3y}\left(8x + 8x^2 - 3y - 6xy + 3y^2\right), \\
f_y &= 3e^{2x+3y}\left(-2x + 8x^2 + 2y - 6xy + 3y^2\right), \\
f_{xx} &= 4e^{2x+3y}\left(4 + 16x + 8x^2 - 6y - 6xy + 3y^2\right), \\
f_{yy} &= 3e^{2x+3y}\left(2 - 12x + 24x^2 + 12y - 18xy + 9y^2\right), \\
f_{xy} &= f_{yx} = 6e^{2x+3y}\left(-1 + 6x + 8x^2 - y - 6xy + 3y^2\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
f_x(0,0) &= 0, \quad f_y(0,0) = 0, \quad f_{xx}(0,0) = 16, \\
f_{yy}(0,0) &= 6, \quad f_{xy}(0,0) = f_{yx}(0,0) = -6,
\end{aligned}$$

and $f(0,0) = 0$. With this we obtain the following Taylor expansion

$$f(x,y) = 8x^2 + 3y^2 - 6xy.$$

To obtain the expansion in terms of the displacements h and k we only need to set $x = x_0 + h$ and $y = y_0 + k$. Since in this case $x_0 = y_0 = 0$,

$$f(h,k) = 8h^2 + 3k^2 - 6hk.$$

The problem also asks what we can conclude about the nature of the point $(0,0)$. Since both first order derivatives vanish at that point we know that it must be either a maximum, a minimum or a saddle point. To know which one it is we need to compute:

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = (16)(6) - 6^2 = 60 > 0.$$

Since $f_{xx}(0,0) = 16 > 0$ the point is in fact a minimum of the function.

(b) In this case our constraint is

$$\phi(x,y,z) = x^3 + y^3 + z^3 - 1 = 0, \tag{0.1}$$

and the corresponding partial derivatives of f and ϕ are

$$\begin{aligned} f_x &= zy, \qquad f_y = xz, \qquad f_z = xy, \\ \phi_x &= 3x^2, \qquad \phi_y = 3y^2, \qquad \phi_z = 3z^2. \end{aligned}$$

Therefore we need to solve the following system of equations

$$\begin{aligned} x^3 + y^3 + z^3 - 1 &= 0, \\ zy + \lambda 3x^2 &= 0, \\ zx + \lambda 3y^2 &= 0, \\ xy + \lambda 3z^2 &= 0. \end{aligned}$$

The last three equations are solved by $x = y = z$ and $\lambda = -1/3$, which when plugged into the first equation gives the condition

$$3x^3 = 1 \Rightarrow x = \sqrt[3]{\frac{1}{3}}.$$

In addition, the equations admit also the solutions $(0,0,1)$, $(1,0,0)$ and $(0,1,0)$ with $\lambda = 0$. At these points $f = 0$ and this is the minimum value of this function for points satisfying (0.1) and $x,y,z \geq 0$. The maximum value of f subject to (0.1) is therefore $1/3$.

3. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 - 4m + 5 = 0 \Rightarrow m = 2 \pm i.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x,$$

therefore we identify

$$u_1(x) = e^{2x} \cos x, \qquad u_2(x) = e^{2x} \sin x.$$

For the second part of the problem we will need the Wronskian of these solutions which is

$$\begin{aligned} W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x}\cos x & e^{2x}\sin x \\ 2e^{2x}\cos x - e^{2x}\sin x & 2e^{2x}\sin x + e^{2x}\cos x \end{vmatrix} \\ &= e^{4x}\cos x(2\sin x + \cos x) - e^{4x}\sin x(2\cos x - \sin x) = e^{4x}. \end{aligned}$$

Therefore the Wronskian is indeed nowhere zero for finite values of x.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = \frac{e^{2x}}{\sin x}, \qquad W(x) = e^{4x},$$

therefore

$$\begin{aligned} v_1(x) &= -\int dx = -x, \\ v_2(x) &= \int \frac{\cos x}{\sin x}dx = \ln|\sin x|. \end{aligned}$$

Hence the general solution of the inhomogeneous equation is

$$y = e^{2x}(c_1\cos x + c_2\sin x - x\cos x + \ln|\sin x|\sin x),$$

with c_1, c_2 being arbitrary constants.

4. Calling

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

and plugging y'' into the differential equation we obtain,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0,$$

which can be rewritten as

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=2}^{\infty} a_{k-2}x^k = 0,$$

by introducing $k = n - 2$ in the first sum and $k = n + 2$ in the second sum. Putting terms of the same order in x together we obtain

$$2a_2 + 6a_3x + \sum_{k=2}^{\infty} \left((k+2)(k+1)a_{k+2} + a_{k-2} \right) x^k = 0,$$

comparing terms with the same power of x we obtain

$$a_2 = a_3 = 0,$$

and for $k = 2, 3, \ldots$

$$(k+2)(k+1)a_{k+2} + a_{k-2} = 0 \qquad \Leftrightarrow \qquad a_{k+2} = -\frac{a_{k-2}}{(k+2)(k+1)}.$$

Now we just have to use the formula to determine the first 4 non-vanishing terms in the y-series. We find

$$k = 2: \; a_4 = -\frac{a_0}{12},$$
$$k = 3: \; a_5 = -\frac{a_1}{20}.$$

We have seen above that a_2 and a_3 vanish, therefore also a_6 and a_7 will vanish. Thus the 4 first non-vanishing terms in the series will be

$$y = a_0 + a_1x - \frac{a_0}{12}x^4 - \frac{a_1}{20}x^5 + \cdots$$

and in order to fix a_0 and a_1 we will need two initial conditions. For example, if $y(0) = y'(0) = 1$, then we would have

$$y(0) = a_0 = 1,$$

and

$$y'(0) = a_1 = 1.$$