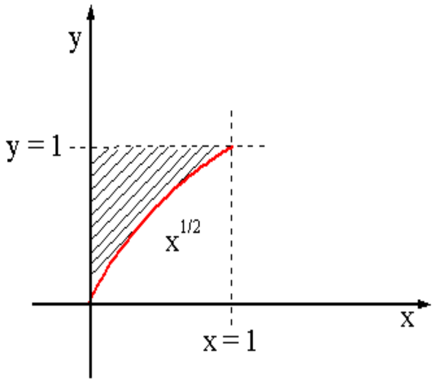
CALCULUS 2001: EXAM SOLUTIONS

1. The integration region is the shaded area in the picture below



From the picture it is easy to see that changing the order of integration we obtain the integral

$$I = \int_{y=0}^{y=1} dy \int_{x=0}^{x=y^2} \sin\left(\frac{y^3+1}{2}\right) dx.$$

The integral in x gives

$$\int_{x=0}^{x=y^2} \sin\left(\frac{y^3+1}{2}\right) dx = \left[x \sin\left(\frac{y^3+1}{2}\right)\right]_0^{y^2} = y^2 \sin\left(\frac{y^3+1}{2}\right).$$

Plugging this result into the second integral we obtain

$$I = \int_{y=0}^{y=1} y^2 \sin\left(\frac{y^3+1}{2}\right) dy = \frac{2}{3}\int_{t=1/2}^{t=1} \sin(t) dt = -\frac{2}{3}\left[\cos(t)\right]_{1/2}^{1} = -\frac{2}{3}(\cos(1) - \cos(1/2)).$$

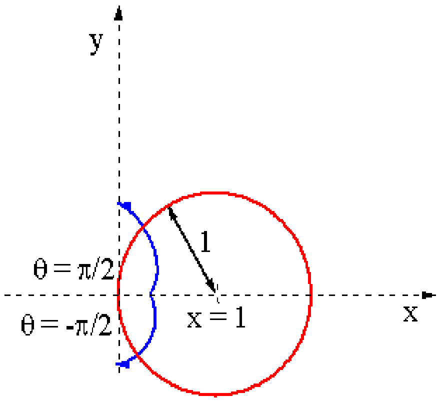
where we introduced the change of variables

$$t = \frac{y^3+1}{2}, \qquad dt = \frac{3}{2}y^2 dy.$$

(b) The circle

$$(x-1)^2 + y^2 = 1,$$

is a circle of radius 1 centered at the point $(1,0)$, in the x-axis. That is the circle in the figure below,



In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Substituting this into the equation of the circle we obtain

$$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1, \quad \Leftrightarrow \quad r^2 - 2r\cos\theta = 0$$

where we have used the identity $\cos^2\theta + \sin^2\theta = 1$. We can cancel a factor r in the equation and we obtain

$$2\cos\theta = r.$$

In addition, it is easy to see from the picture above that the angle θ varies between $-\pi/2$ and $\pi/2$.

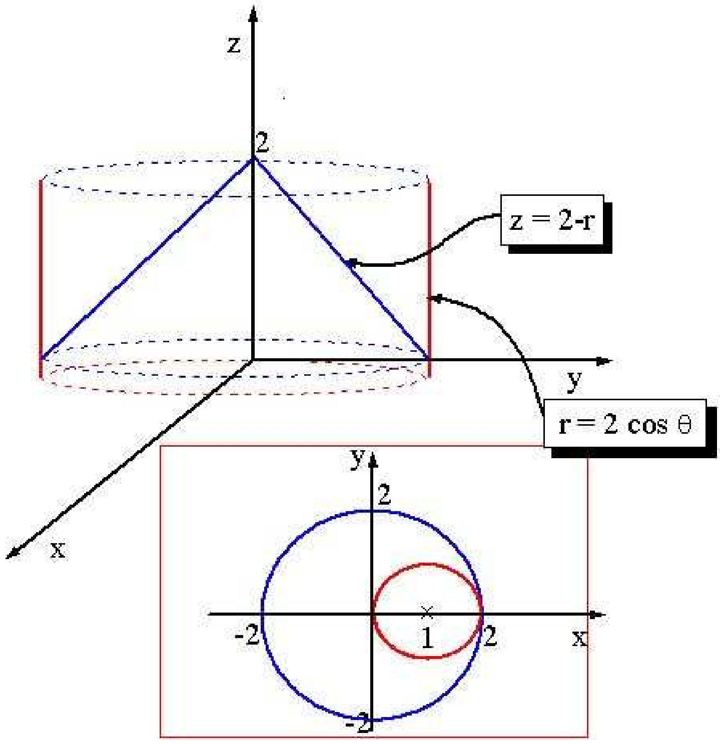
In the second part of the exercise we need to compute a volume by performing a 3-dimensional integral on the region bounded by the cylinder

$$(x-1)^2 + y^2 = 1 \quad \Leftrightarrow \quad r = 2\cos\theta,$$

and the cone

$$z = 2 - \sqrt{x^2 + y^2} \quad \Leftrightarrow \quad z = 2 - r,$$

where we have already written the corresponding equations in cylindrical coordinates. The equation of the cylinder is exactly the same as for the circle above, which means that we can sketch the cylinder by shifting the circle above along the z direction. The integration region will be roughly as sketched below.



For the 3D picture is actually not so easy to see that the cone and the cylinder are not both centered at the z-axis. This is only the case for the cone (which is centered at $(0,0,2)$). The cylinder is centered at $x = 1$. So, if we would take a look at our cylinder and cone from above, we should see something like the figure in the lower corner of the picture above. The integration region is the volume which is both inside the cone and insider the cylinder. In cylindrical coordinates this region is described as

$$R = \{(r,\theta,z) \mid 0 \leq r \leq 2\cos\theta,\ -\pi/2 \leq \theta \leq \pi/2,\ 0 \leq z \leq 2-r\}.$$

The element of volume in cylindrical coordinates is

$$dx\,dy\,dz = r\,d\theta\,dr\,dz,$$

therefore the volume we have to compute is

$$V = \int_{\theta=-\pi/2}^{\theta=\pi/2} d\theta \int_{r=0}^{r=2\cos\theta} rdr \int_{z=0}^{z=2-r} dz.$$

The z-integral is easily carried out

$$\int_{z=0}^{z=2-r} dz = [z]_{z=0}^{z=2-r} = 2-r.$$

Substituting this result into the r-integral we obtain

$$\int_{r=0}^{r=2\cos\theta} r(2-r)dr = \left[r^2 - r^3/3\right]_{r=0}^{r=2\cos\theta} = 4\cos^2\theta - \frac{8\cos^3\theta}{3} = \frac{4}{3}\cos^2\theta(3-2\cos\theta).$$

Finally

$$V = \frac{4}{3}\int_{\theta=-\pi/2}^{\theta=\pi/2} \cos^2\theta(3-2\cos\theta)d\theta = \frac{4}{3}\int_{\theta=-\pi/2}^{\theta=\pi/2}(3\cos^2\theta - 2\cos\theta(1-\sin^2\theta))d\theta$$
$$= \frac{4}{3}\int_{\theta=-\pi/2}^{\theta=\pi/2}\left(\frac{3}{2}(\cos(2\theta)+1) - 2\cos\theta(1-\sin^2\theta)\right)d\theta$$
$$= \frac{4}{3}\int_{\theta=-\pi/2}^{\theta=\pi/2}\left(\frac{3\cos(2\theta)}{2} + \frac{3}{2} - 2\cos\theta + 2\cos\theta\sin^2\theta\right)d\theta$$
$$= \frac{4}{3}\left[\frac{3\sin(2\theta)}{4} + \frac{3\theta}{2} - 2\sin\theta + \frac{2}{3}\sin^3\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2} = 2\pi - \frac{32}{9}.$$

2. (a) As usual we commence by finding the points at which the 1st-order partial derivatives are zero. In this case we have

$$f_x = -y(1-x^2)e^{-(x^2+y^2)/2}, \qquad f_y = -x(1-y^2)e^{-(x^2+y^2)/2}.$$

We find that $f_x = 0$ for $x = \pm 1$ and $y = 0$ and $f_y = 0$ for $y = \pm 1$ and $x = 0$. Therefore the two partial derivatives vanish at the following 5 points:

$$(1,1) \quad (1,-1) \quad (-1,1) \quad (-1,-1) \quad (0,0).$$

We compute now the 2nd-order partial derivatives:

$$\begin{aligned} f_{xx} &= -xy(x^2-3)e^{-(x^2+y^2)/2}, \quad f_{yy} = -xy(y^2-3)e^{-(x^2+y^2)/2}, \\ f_{xy} &= f_{yx} = -(1-x^2)(1-y^2)e^{-(x^2+y^2)/2}, \end{aligned}$$

and study each of the points separately:

The point $(1,1)$: At this point

$$A = 2/e, \qquad B = 0, \qquad C = 2/e.$$

Therefore $A > 0$ and $AC - B^2 = 4/e^2 > 0$. This point is a minimum.

The point $(-1,-1)$: At this point

$$A = 2/e, \qquad B = 0, \qquad C = 2/e.$$

Therefore $A > 0$ and $AC - B^2 = 4/e^2 > 0$. This point is also a minimum.

The point $(1,-1)$: At this point

$$A = -2/e, \qquad B = 0, \qquad C = -2/e.$$

Therefore $A < 0$ and $AC - B^2 = 4/e^2 > 0$. This point is a maximum.

The point $(-1,1)$: At this point

$$A = -2/e, \qquad B = 0, \qquad C = -2/e.$$

Therefore $A < 0$ and $AC - B^2 = 4/e^2 > 0$. This point is also a maximum.

The point $(0,0)$: At this point

$$A = 0, \qquad B = 1, \qquad C = 0.$$

Therefore $A = 0$ and $B \neq 0$. This point is a saddle point.

Note: Notice that the problem only asks to classify the point $(0,0)$, so it would suffice to show that $(0,0)$ is a saddle point.

(b)The Taylor expansion of a function of two variables $f(x,y)$ around a point (x_0,y_0) up to second order terms is given by

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0,y_0) = (1,1)$ and

$$\begin{aligned} f_x &= 3x^2y^3, \quad f_y = 3x^3y^2, \quad f_{xx} = 6xy^3, \\ f_{yy} &= 6x^3y, \qquad f_{xy} = f_{yx} = 9x^2y^2. \end{aligned}$$

Therefore

$$\begin{aligned} f_x(1,1) &= 3, \quad f_y(1,1) = 3, \quad f_{xx}(1,1) = 6, \\ f_{yy}(1,1) &= 6, \qquad f_{xy}(1,1) = f_{yx}(1,1) = 9, \end{aligned}$$

and $f(1,1) = 1$. With this we obtain the following Taylor expansion

$$\begin{aligned} f(x,y) &= 1 + 3(x-1) + 3(y-1) + 3(x-1)^2 + 3(y-1)^2 + 9(x-1)(y-1) \\ &= 10 + 3(x^2+y^2) - 12(x+y) + 9xy. \end{aligned}$$

Substituting $x = 1+h$ and $y = 1+k$ above we obtain the corresponding expansion on the displacements

$$f(1+h, 1+k) = 1 + 3(h^2 + k^2 + h + k) + 9hk.$$

Setting $x = 1+h$ and $k = 1+h$ in the original function, we obtain

$$\begin{aligned} f(x,y) &= (1+h)^3(1+k)^3 = 1 + 3(h^2+k^2+h+k) + 9hk \\ &+ h^3 + 9h^2k + 3h^3k + 9hk^2 + 9h^2k^2 + 3h^3k^2 + k^3 + 3hk^3 + 3h^2k^3 + h^3k^3, \end{aligned}$$

and so we see directly that the terms containing only powers 2 or less of h, k are those obtained by the Taylor expansion.

3. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 + 2m + 2 = 0 \Rightarrow m = -1 \pm i.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x,$$

therefore we identify

$$u_1(x) = e^{-x}\cos x, \qquad u_2(x) = e^{-x}\sin x.$$

For the second part of the problem we will need the Wronskian of these solutions which is

$$\begin{aligned} W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ -e^{-x}\cos x - e^{-x}\sin x & -e^{-x}\sin x + e^{-x}\cos x \end{vmatrix} \\ &= e^{-2x}\cos x(-\sin x + \cos x) - e^{-2x}\sin x(-\cos x - \sin x) = e^{-2x}. \end{aligned}$$

Therefore the Wronskian is indeed nowhere zero for finite values of x.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = \frac{e^{-x}}{\cos^3 x}, \qquad W(x) = e^{-2x},$$

therefore

$$\begin{aligned} v_1(x) &= -\int \frac{dx}{\cos^2 x} = -\tan x, \\ v_2(x) &= \int \frac{\sin x}{\cos^3 x}dx = \int \frac{\tan x}{\cos^2 x}dx = \frac{1}{2}\tan^2 x. \end{aligned}$$

The second integral can also be performed by using the change of variables $t = \tan x$. Hence the general solution of the inhomogeneous equation is

$$y = e^{-x}(c_1\cos x + c_2\sin x - \tan x\cos x + \frac{1}{2}\tan^2 x\sin x),$$

with c_1, c_2 being arbitrary constants.

4. (a) In this case our constraint is

$$\phi(x,y) = y^2 + x^2 + y^3 + x^3 = 0, \tag{0.1}$$

and the function we want to minimize is the distance from the point $(0,0,1)$ to a point $(x,y,0)$ in the curve above. The square of the distance is the function

$$f(x,y) = x^2 + y^2 + 1,$$

and the key thing to notice in the problem is that the curve lies on the xy-plane and therefore the point which is closest to $(0,0,1)$ and lies in the curve (0.1) has coordinate $z = 0$. This means that we have a problem of Lagrange multipliers but we only have equations in x and y. The corresponding partial derivatives of f and ϕ are

$$\begin{aligned} f_x &= 2x, \qquad f_y = 2y, \\ \phi_x &= 2x + 3x^2, \qquad \phi_y = 2y + 3y^2. \end{aligned}$$

Therefore we need to solve the following system of equations

$$\begin{aligned} y^2 + x^2 + y^3 + x^3 = 0 &= 0, \\ 2x + \lambda(2+3x)x &= 0, \\ 2y + \lambda(2+3y)y &= 0. \end{aligned}$$

It is easy to see that $x = y = 0$ solves all equations above, so one potential solution of the problem is the point $(0,0,0)$. If $x, y \neq 0$ then from the last two equations we have

$$\lambda = -\frac{1}{2+3x} = -\frac{1}{2+3y},$$

and from this equality we obtain

$$2 + 3x = 2 + 3y \quad \Rightarrow \quad x = y.$$

Substituting $x = y$ into the constraint (0.1) we obtain

$$2x^2 + 2x^3 = 0 \quad \Rightarrow \quad 2x^2(1+x) = 0 \quad \Rightarrow \quad x = y = 0, -1.$$

For $x = y = -1$ we obtain $\lambda = 1$. Therefore the only solutions to the problem are

$$x = y = -1, \qquad x = y = 0.$$

and substituting them into the square distance $f(x,y)$ we see that they give us

$$d = \sqrt{f(0,0)} = 1,$$

and

$$d = \sqrt{f(-1,-1)} = \sqrt{3}.$$

Therefore the point $(0,0,0)$ is the point of the curve (0.1) which is closest to $(0,0,1)$ and the point $(-1,-1,0)$ is the point of the curve which is farthest from $(0,0,1)$. (b)Let us consider an implicit function of two variables $z = f(x,y)$ and assume the existence of a constraint

$$F(x,y,z) = 0,$$

which relates the function z to the two independent variables x and y. Since $F = 0$ it is clear that also its total differential $dF = 0$ must vanish. However the total differential is by definition

$$dF = \left(\frac{\partial F}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y}\right) dy + \left(\frac{\partial F}{\partial z}\right) dz = 0, \tag{0.2}$$

and in addition, z is a function of x and y, therefore its differential is given by

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy. \tag{0.3}$$

If we substitute (0.3) into (0.2) we obtain the equation

$$dF = 0 = \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\right) dy. \tag{0.4}$$

Since x and y are independent variables, equation (0.4) implies that each of the factors has to vanish separately, that is

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0. \tag{0.5}$$

Therefore we obtain,

$$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{F_x}{F_z},$$

$$\frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial F}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{F_y}{F_z}.$$

Employing now these formulae for the function $F(x,y,z) = z - xy^2z^3 - 2xyz = 0$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{y^2z^3 + 2yz}{1 - 3xy^2z^2 - 2xy},$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{2yz^3 + 2xz}{1 - 3xy^2z^2 - 2xy}.$$

If $z - ax + by + c = 0$ is tangent to the surface $F(x,y,z) = z - xy^2z^3 - 2xyz = 0$ at $(-1,-1,1)$ this just means that

$$-a = \left.\frac{\partial z}{\partial x}\right|_{(-1,-1,1)} = -\frac{1}{2},$$

and

$$b = \left.\frac{\partial z}{\partial y}\right|_{(-1,-1,1)} = -\frac{4}{2} = -2.$$

So the equation of the plane is

$$z - x/2 - 2y + c = 0,$$

and the value of c is fixed by imposing that the point $(-1,-1,1)$ is contained on the plane

$$1 + 1/2 + 2 + c = 0 \quad \Leftrightarrow \quad c = -7/2.$$

Therefore the tangent plane is

$$z - x/2 - 2y - 7/2 = 0.$$