CALCULUS 2002: EXAM SOLUTIONS

1. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 - 4m + 5 = 0 \Rightarrow m = 2 \pm i.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x,$$

therefore we identify

$$u_1(x) = e^{2x} \cos x, \qquad u_2(x) = e^{2x} \sin x.$$

For the second part of the problem we will need the Wronskian of these solutions which is

$$\begin{aligned} W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} \cos x & e^{2x} \sin x \\ 2e^{2x} \cos x - e^{2x} \sin x & 2e^{2x} \sin x + e^{2x} \cos x \end{vmatrix} \\ &= e^{4x} \cos x (2 \sin x + \cos x) - e^{4x} \sin x (2 \cos x - \sin x) = e^{4x}. \end{aligned}$$

Therefore the Wronskian is indeed nowhere zero for finite values of x.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x) u_1(x) + v_2(x) u_2(x),$$

with

$$v_1(x) = -\int u_2(x) \frac{R(x)}{W(x)} dx \qquad \text{and} \qquad v_2(x) = \int u_1(x) \frac{R(x)}{W(x)} dx.$$

In our case

$$R(x) = \frac{2e^{2x}}{\sin x}, \qquad W(x) = e^{4x},$$

therefore

$$\begin{aligned} v_1(x) &= -2 \int dx = -2x, \\ v_2(x) &= 2 \int \frac{\cos x}{\sin x} dx = 2 \ln |\sin x|. \end{aligned}$$

Hence the general solution of the inhomogeneous equation is

$$y = e^{2x} (c_1 \cos x + c_2 \sin x - 2x \cos x + 2 \ln(\sin x) \sin x),$$

with c_1, c_2 being arbitrary constants.

2. Calling

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

and plugging y'' into the differential equation we obtain,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0,$$

which can be rewritten as

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=2}^{\infty} a_{k-2} x^k = 0,$$

by introducing $k = n-2$ in the first sum and $k = n+2$ in the second sum. Putting terms of the same order in x together we obtain

$$2a_2 + 6a_3 x + \sum_{k=2}^{\infty} \left((k+2)(k+1) a_{k+2} + a_{k-2}\right) x^k = 0,$$

comparing terms with the same power of x we obtain

$$a_2 = a_3 = 0,$$

and for $k = 2, 3, \ldots$

$$(k+2)(k+1) a_{k+2} + a_{k-2} = 0 \qquad \Leftrightarrow \qquad a_{k+2} = -\frac{a_{k-2}}{(k+2)(k+1)}.$$

Now we just have to use the formula to determine the first 4 non-vanishing terms in the y-series. We find

$$k = 2 : \ a_4 = -\frac{a_0}{12},$$
$$k = 3 : \ a_5 = -\frac{a_1}{20}.$$

We have seen above that a_2 and a_3 vanish, therefore also a_6 and a_7 will vanish. Thus the 4 first non-vanishing terms in the series will be

$$y = a_0 + a_1 x - \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 + \cdots$$

and in order to fix a_0 and a_1 we will need two initial conditions. For example, if $y(0) = y'(0) = 1$, then we would have

$$y(0) = a_0 = 1,$$

and

$$y'(0) = a_1 = 1.$$

3. (a) The integration region is the triangle formed by the intersection of the lines $y = x$, $y = 0$ and $x = 1$. Once we have identified the integration region, it is easy to change the order of integration to write I equivalently as

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=x} \cos\left(\frac{\pi x^2}{2}\right) dy.$$

The integral

$$\int_{y=0}^{y=x} \cos\left(\frac{\pi x^2}{2}\right) dy = \left[y \cos\left(\frac{\pi x^2}{2}\right)\right]_0^x = x\cos\left(\frac{\pi x^2}{2}\right) - 0 = x\cos\left(\frac{\pi x^2}{2}\right),$$

is trivial to do, since the argument does not depend on y. Now the second integral is also very easy to do, since we have the product of the cosine of a function and the derivative of that function, therefore

$$I = \int_{x=0}^{x=1} x\cos\left(\frac{\pi x^2}{2}\right) dx = \left[\frac{1}{\pi}\sin\left(\frac{\pi x^2}{2}\right)\right]_0^1 = \frac{1}{\pi} - 0 = \frac{1}{\pi}.$$

If you do not realize how to do the integral directly, you can also change variables to $t = \pi x^2/2$ which gives $dt = \pi x dx$ and allows you to rewrite the integral above as

$$I = \int_{x=0}^{x=1} x\cos\left(\frac{\pi x^2}{2}\right) dx = \frac{1}{\pi}\int_{t=0}^{t=\pi/2} \cos(t)\, dt = \frac{1}{\pi}\left[\sin(t)\right]_0^{\pi/2} = \frac{1}{\pi}.$$

(b) We start by computing the Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -v/u^2 & 1/u \\ 0 & 1 \end{vmatrix} = -\frac{v}{u^2}.$$

Therefore

$$dx\, dy = |J| du\, dv = \frac{v}{u^2} du\, dv.$$

Now we have to transform the function we want to integrate,

$$\frac{y^2}{x^2} e^{y/x} = u^2 e^u,$$

and we have to find the new integration region

$$0 \le x \le 1 \quad \Leftrightarrow \quad 0 \le v \le u,$$
$$0 \le y \le x \quad \Leftrightarrow \quad 0 \le u \le 1.$$

Therefore the integral we need to compute is

$$I = \int_{u=0}^{u=1} e^u du \int_{v=0}^{v=u} v\, dv.$$

The first integral is

$$\int_{v=0}^{v=u} v\,dv = \left[\frac{v^2}{2}\right]_{v=0}^{v=u} = \frac{u^2}{2},$$

and so

$$I = \frac{1}{2}\int_{u=0}^{u=1} u^2 e^u du.$$

This integral can be done by using integration by parts twice

$$\begin{aligned}\int_{u=0}^{u=1} u^2 e^u du &= \left[u^2 e^u\right]_0^1 - \int_{u=0}^{u=1} 2ue^u du = e - \int_{u=0}^{u=1} 2ue^u du \\ &= e - [2ue^u]_0^1 + \int_{u=0}^{u=1} 2e^u du = e - 2e + \int_{u=0}^{u=1} 2e^u du \\ &= [2e^u]_0^1 - e = 2e - 2 - e = e - 2.\end{aligned}$$

Therefore

$$I = \frac{e-2}{2}.$$

4. (a) The Taylor expansion of a function of two variables $f(x,y)$ around a point (x_0, y_0) up to second order terms is given by

$$\begin{aligned}f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0).\end{aligned}$$

First we need to compute the 1st and 2nd order partial derivatives

$$\begin{aligned}f_x &= 2e^{2x+3y}\left(8x + 8x^2 - 3y - 6xy + 3y^2\right), \\ f_y &= 3e^{2x+3y}\left(-2x + 8x^2 + 2y - 6xy + 3y^2\right), \\ f_{xx} &= 4e^{2x+3y}\left(4 + 16x + 8x^2 - 6y - 6xy + 3y^2\right), \\ f_{yy} &= 3e^{2x+3y}\left(2 - 12x + 24x^2 + 12y - 18xy + 9y^2\right), \\ f_{xy} &= f_{yx} = 6e^{2x+3y}\left(-1 + 6x + 8x^2 - y - 6xy + 3y^2\right).\end{aligned}$$

Therefore

$$\begin{aligned}f_x(0,0) &= 0, \quad f_y(0,0) = 0, \quad f_{xx}(0,0) = 16, \\ f_{yy}(0,0) &= 6, \quad f_{xy}(0,0) = f_{yx}(0,0) = -6,\end{aligned}$$

and $f(0,0) = 0$. With this we obtain the following Taylor expansion

$$f(x,y) = 8x^2 + 3y^2 - 6xy.$$

To obtain the expansion in terms of the displacements h and k we only need to set $x = x_0 + h$ and $y = y_0 + k$. Since in this case $x_0 = y_0 = 0$,

$$f(h,k) = 8h^2 + 3k^2 - 6hk.$$

The problem also asks what we can conclude about the nature of the point $(0,0)$. Since both first order derivatives vanish at that point we know that it must be either a maximum, a minimum or a saddle point. To know which one it is we need to compute:

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = (16)(6) - 6^2 = 60 > 0.$$

Since $f_{xx}(0,0) = 16 > 0$ the point is in fact a minimum of the function.

(b) In this case our constraint is

$$\phi(x,y) = y^2 + x^2 + 4xy - 4 = 0, \tag{0.1}$$

and the function we want to minimize is the distance from the point $(0,0,1)$ to a point $(x,y,0)$ in the curve above. The square of the distance is the function

$$f(x,y) = x^2 + y^2 + 1,$$

and the key thing to notice in the problem is that the curve lies on the xy-plane and therefore the point which is closest to $(0,0,1)$ and lies in the curve (0.1) has coordinate $z = 0$. This means that we have a problem of Lagrange multipliers but we only have equations in x and y. The corresponding partial derivatives of f and ϕ are

$$\begin{aligned} f_x &= 2x, \qquad f_y = 2y, \\ \phi_x &= 2x + 4y, \qquad \phi_y = 2y + 4x. \end{aligned}$$

Therefore we need to solve the following system of equations

$$\begin{aligned} y^2 + x^2 + 4xy - 4 = 0 &= 0, \\ 2x + \lambda(2x + 4y) &= 0, \\ 2y + \lambda(2y + 4x) &= 0. \end{aligned}$$

The last two equations give

$$\lambda = -\frac{x}{x+2y} = -\frac{y}{y+2x}.$$

and from this equality we obtain

$$x(y+2x) = y(x+2y) \quad \Rightarrow \quad x^2 = y^2 \quad \Rightarrow \quad x = \pm y.$$

For $x = y$ we obtain $\lambda = -1/2$ and for $x = -y$ we have $\lambda = 1$. Substituting $x = y$ into the constraint (0.1) we obtain

$$6x^2 - 4 = 0 \quad \Rightarrow \quad x^2 = \frac{2}{3} \quad \Rightarrow \quad x = y = \pm\sqrt{\frac{2}{3}}.$$

Taking now the other solution $x = -y$ and substituting it into (0.1) we obtain

$$-2x^2 - 4 = 0 \quad \Rightarrow \quad x^2 = -2,$$

and this solution does not make sense, since it gives x, y imaginary. Therefore the only sensible solutions to the problem are

$$x = y = \sqrt{\frac{2}{3}}, \qquad x = y = -\sqrt{\frac{2}{3}}.$$

and substituting them into the square distance $f(x, y)$ we see that they give us the same distance

$$d = \sqrt{f(\pm\sqrt{\frac{2}{3}}, \pm\sqrt{\frac{2}{3}})} = \sqrt{\frac{7}{3}}.$$

Therefore there are two points contained in the curve (0.1) which are both at the same distance from $(0, 0, 1)$ and this is also the shortest distance.