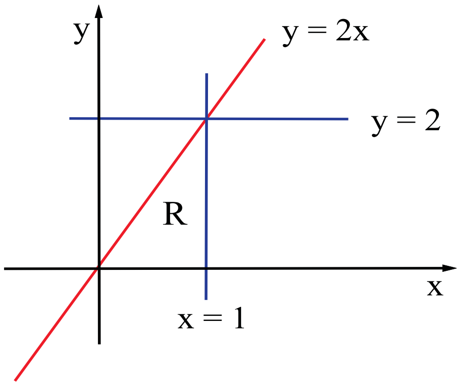
CALCULUS 2004: EXAM SOLUTIONS

1. (a) The integration region is the lower triangle in the picture



From the picture it is easy to see that changing the order of integration we obtain the integral

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=2x} e^{x^2} dy.$$

The integral in y gives

$$\int_{y=0}^{y=2x} e^{x^2} dy = \left[y e^{x^2} \right]_0^{2x} = 2x e^{x^2}.$$

Plugging this result into the second integral we obtain

$$I = \int_{x=0}^{x=1} 2x e^{x^2} dx = \left[e^{x^2} \right]_0^1 = e - 1.$$

(b) The polar coordinates (r, θ) are related to the Cartesian coordinates by

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

and therefore the equation of the semi-circle becomes (since r is always positive)

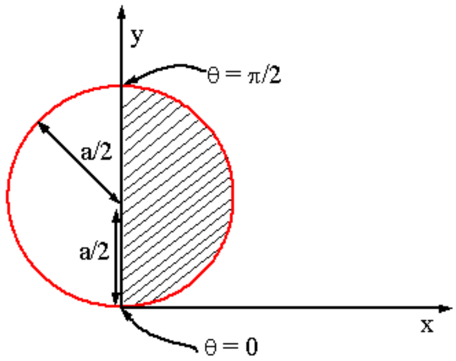
$$r^2 - ar\sin\theta = 0 \quad \Leftrightarrow \quad r = a\sin\theta.$$

In order to determine in which region θ takes its values it is convenient to rewrite the equation of the circle as

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4},$$

in this form we can directly see that the equation describes a circle centered at the point $(0, a/2)$ of radius $a/2$. That is the circle depicted below, were the dashed region is the semi-circle having $x \leq 0$. For any point contained on that semi-circle the angle θ takes values

$$0 \leq \theta \leq \pi/2.$$



In the second part of the exercise we need to compute the volume

$$V = \int\int\int_R dxdydz,$$

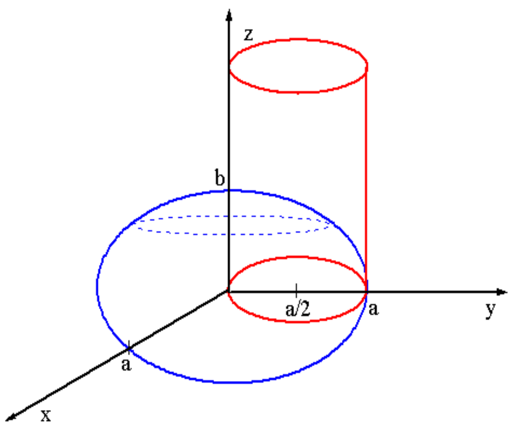
where the integration region R is determined by the intersection of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1,$$

and the cylinder

$$x^2 + y^2 - ay = 0, \qquad x \leq 0.$$

The last equation is the same as for the semi-circle in the last figure. This means that the cylinder is generated by shifting the circle above along the z axis. Since $x \leq 0$ in fact only half of the cylinder has to be considered. The cylinder and the ellipsoid are depicted below



In cylindrical coordinates, the equation of the ellipsoid becomes

$$\frac{r^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \Leftrightarrow \quad z = \pm b\sqrt{1 - \frac{r^2}{a^2}},$$

whereas the equation of the cylinder was already obtained above

$$r = a \sin\theta, \quad 0 \le \theta \le \pi/2.$$

The problem tells us also that we must only considered the region above the xy-plane, that is $z \ge 0$. The integration region in cylindrical coordinates is then

$$R = \left\{ (r,\theta,z) \,|\, 0 \le r \le a\sin\theta, \quad 0 \le \theta \le \pi/2, \quad 0 \le z \le b\sqrt{1 - \frac{r^2}{a^2}} \right\},$$

The element of volume in cylindrical coordinates is

$$dx\,dy\,dz = r\,dr\,d\theta\,dz,$$

and therefore the volume is

$$V = \int_{\theta=0}^{\theta=\pi/2} d\theta \int_{r=0}^{r=a\sin\theta} r\,dr \int_{z=0}^{z=b\sqrt{1-r^2/a^2}} dz.$$

The integral in z gives

$$\int_{z=0}^{z=b\sqrt{1-r^2/a^2}} dz = b\sqrt{1 - r^2/a^2}.$$

Plugging this into the r-integral we have

$$\int_{r=0}^{r=a\sin\theta} rb\sqrt{1 - r^2/a^2} dr,$$

changing variables to $t = 1 - r^2/a^2$ we obtain

$$dt = -2r/a^2 dr.$$

The integration limit $r = 0$ corresponds to $t = 1$ and $r = a\sin\theta$ corresponds to $t = 1 - \sin^2\theta = \cos^2\theta$, so that the integral becomes

$$-\frac{a^2 b}{2}\int_{t=1}^{t=\cos^2\theta} t^{1/2} dt = -\frac{a^2 b}{3}\left[t^{3/2}\right]_{t=1}^{t=\cos^2\theta} = -\frac{a^2 b}{3}(\cos^3\theta - 1).$$

Substituting this result into the θ-integral we obtain

$$\begin{aligned} V &= \frac{a^2 b}{3}\int_{\theta=0}^{\theta=\pi/2}(1 - \cos^3\theta)d\theta = \frac{a^2 b}{3}\int_{\theta=0}^{\theta=\pi/2}(1 - \cos\theta(1 - \sin^2\theta))d\theta \\ &= \frac{a^2 b}{3}\int_{\theta=0}^{\theta=\pi/2}(1 - \cos\theta + \cos\theta\sin^2\theta)d\theta = \frac{a^2 b}{3}\left[\theta - \sin\theta + \frac{\sin^3\theta}{3}\right]_{\theta=0}^{\theta=\pi/2} \\ &= \frac{a^2 b}{3}\left(\frac{\pi}{2} - 1 + \frac{1}{3}\right) = \frac{a^2 b}{18}(3\pi - 4). \end{aligned}$$

2. (a) For this function the first order partial derivatives are

$$f_x = 3x^2 + y^2 - 24x + 21,$$
$$f_y = 2xy - 4y.$$

The first thing we have to do is finding the points at which these derivatives vanish

$$f_y = 0 \quad \Rightarrow \; y(x-2) = 0 \quad \Rightarrow y = 0 \; \text{ or } \; x = 2.$$

For $y = 0$ (which is one of the solutions of the previous equation) f_x will vanish if

$$f_x(x, y = 0) = 0 = 3x^2 - 24x + 21 = 0 \; \Rightarrow \; x = \frac{24 \pm 18}{6} = 7, 1,$$

and for $x = 2$ (which is the other solution of $f_y = 0$) we would obtain

$$f_x(x = 2, y) = 0 = 12 + y^2 - 48 + 21 \quad \Rightarrow \quad y = \pm\sqrt{15}.$$

Therefore, putting all these solutions together we have the following 4 points:

$$(x, y) = (1, 0), \; (7, 0), \; (2, \sqrt{15}) \quad \text{and} \quad (2, -\sqrt{15}).$$

The next step is to compute the second order partial derivatives

$$A = f_{xx} = 6x - 24,$$
$$B = f_{xy} = f_{yx} = 2y,$$
$$C = f_{yy} = 2x - 4,$$

therefore

$$AC - B^2 = (6x - 24)(2x - 4) - 4y^2,$$

and we have to study the sign of this quantity in order to classify the stationary points of the function:

The point $(1, 0)$**:** At this point

$$AC - B^2 = (6 - 24)(2 - 4) = 36 > 0,$$
$$A = 6 - 24 = -18 < 0,$$

therefore this point is a **maximum**.

The point $(7, 0)$**:** At this point

$$AC - B^2 = (42 - 24)(14 - 4) = 180 > 0,$$
$$A = 42 - 24 = 18 > 0,$$

therefore this point is a **minimum**.

The point $(2, \sqrt{15})$**:** At this point

$$AC - B^2 = (12 - 24)(4 - 4) - 60 = -60 < 0,$$

therefore this point is a **saddle point**.

The point $(2, -\sqrt{15})$**:** At this point

$$AC - B^2 = (12 - 24)(4 - 4) - 60 = -60 < 0,$$

therefore this point is also a **saddle point**.

(b) The Taylor expansion of a function of two variables $f(x, y)$ around a point (x_0, y_0) up to second order terms is given by

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0, y_0) = (-1, -1)$ and

$$\begin{aligned} f_x &= (1+x+y)e^{x-y}, \quad f_y = (1-x-y)e^{x-y}, \quad f_{xx} = (2+x+y)e^{x-y}, \\ f_{yy} &= (-2+x+y)e^{x-y}, \qquad f_{xy} = f_{yx} = -(x+y)e^{x-y}. \end{aligned}$$

Therefore

$$\begin{aligned} f_x(-1,-1) &= -1, \quad f_y(-1,-1) = 3, \quad f_{xx}(-1,-1) = 0, \\ f_{yy}(-1,-1) &= -4, \qquad f_{xy}(-1,-1) = f_{yx}(-1,-1) = 2, \end{aligned}$$

and $f(-1, -1) = -2$. With this we obtain the following Taylor expansion

$$\begin{aligned} f(x,y) &= -2 - (x+1) + 3(y+1) - 2(y+1)^2 + 2(y+1)(x+1) \\ &= y + x - 2y^2 + 2xy. \end{aligned}$$

Therefore, the approximate value of $f(-0.9, -1.05)$ is

$$f(-0.9, -1.05) \simeq -0.9 - 1.05 - 2(1.05)^2 + 2(0.9)(1.05) = -2.265.$$

The Taylor expansion in terms of the displacements h and k is obtained simply by replacing $x = x_0 + h = h - 1$ and $y = y_0 + k = k - 1$ in our final formula. It gives

$$f(h,k) = k + h - 2 - 2(k-1)^2 + 2(k-1)(h-1).$$

3. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 \sin(2x) + c_2 \cos(2x),$$

therefore we identify

$$u_1(x) = \sin(2x), \qquad u_2(x) = \cos(2x).$$

For the second part of the problem we will need the Wronskian of these solutions which is

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} \sin(2x) & \cos(2x) \\ 2\cos(2x) & -2\sin(2x) \end{vmatrix} = -2\sin^2(2x) - 2\cos^2(2x) = -2.$$

Therefore the Wronskian is indeed nowhere zero.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = \tan(2x), \qquad W(x) = -2,$$

therefore

$$v_1(x) = \frac{1}{2}\int \cos(2x)\tan(2x)dx = \frac{1}{2}\int \sin(2x)dx = \frac{1}{4}\cos(2x).$$

$$v_2(x) = -\frac{1}{2}\int \sin(2x)\tan(2x)dx = -\frac{1}{2}\int \frac{\sin^2(2x)}{\cos(2x)}dx.$$

A possible way of doing this integral is to change variables as

$$t = \sin(2x), \qquad dt = 2\cos(2x)dx.$$

The integral in the new variables becomes

$$\begin{aligned} v_2(x) &= -\frac{1}{4}\int \frac{t^2}{1-t^2}dt = \frac{1}{4}\int \frac{1-t^2-1}{1-t^2}dt = \frac{1}{4}\int \left(1 - \frac{1}{1-t^2}\right)dt = \frac{t}{4} - \frac{1}{4}\int \frac{1}{1-t^2}dt \\ &= \frac{t}{4} - \frac{1}{8}\int \left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt = \frac{t}{4} - \frac{1}{8}\left(-\ln|1-t| + \ln|1+t|\right) = \frac{t}{4} - \frac{1}{8}\ln\left|\frac{1+t}{1-t}\right| \\ &= \frac{\sin(2x)}{4} - \frac{1}{8}\ln\left|\frac{1+\sin(2x)}{1-\sin(2x)}\right|. \end{aligned}$$

Hence the general solution of the inhomogeneous equation is

$$y = c_1\sin(2x) + c_2\cos(2x) + \frac{1}{2}\sin(2x)\cos(2x) - \frac{\cos(2x)}{8}\ln\left|\frac{1+\sin(2x)}{1-\sin(2x)}\right|$$

with c_1, c_2 being arbitrary constants.

4. (a) Here we simply have to use the chain rule

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta f_x + \sin\theta f_y,$$

and

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta f_x + r\cos\theta f_y.$$

(b) In order to prove the identity we need to compute the second order partial derivatives for a function $f = V$ using the results of part (a). We obtain

$$\begin{aligned}
V_{rr} &= \frac{\partial}{\partial r}(\cos\theta V_x + \sin\theta V_y) = \cos\theta\frac{\partial V_x}{\partial r} + \sin\theta\frac{\partial V_y}{\partial r} \\
&= \cos\theta(\cos\theta V_{xx} + \sin\theta V_{yx}) + \sin\theta(\cos\theta V_{xy} + \sin\theta V_{yy}), \\
&= \cos^2\theta V_{xx} + \sin^2\theta V_{yy} + 2\sin\theta\cos\theta V_{xy}, \\
V_{\theta\theta} &= \frac{\partial}{\partial\theta}(-r\sin\theta V_x + r\cos\theta V_y) \\
&= -r\cos\theta V_x - r\sin\theta\frac{\partial V_x}{\partial\theta} - r\sin\theta V_y + r\cos\theta\frac{\partial V_y}{\partial\theta} \\
&= -r\cos\theta V_x - r\sin\theta V_y - r\sin\theta(-r\sin\theta V_{xx} + r\cos\theta V_{yx}) \\
&+ r\cos\theta(-r\sin\theta V_{xy} + r\cos\theta V_{yy}) = -r\cos\theta V_x - r\sin\theta V_y \\
&+ r^2\sin^2\theta V_{xx} + r^2\cos^2\theta V_{yy} - 2r^2\sin\theta\cos\theta V_{xy},
\end{aligned}$$

and therefore

$$\begin{aligned}
V_{rr} + \frac{1}{r^2}V_{\theta\theta} + \frac{1}{r}V_r &= \cos\theta(\cos\theta V_{xx} + \sin\theta V_{yx}) + \sin\theta(\cos\theta V_{xy} + \sin\theta V_{yy}) \\
&+ \frac{1}{r^2}(-r\cos\theta V_x - r\sin\theta V_y + r^2\sin^2\theta V_{xx} + r^2\cos^2\theta V_{yy} - 2r^2\sin\theta\cos\theta V_{xy}) \\
&+ \frac{1}{r}(\cos\theta V_x + \sin\theta V_y) = V_{xx} + V_{yy}.
\end{aligned}$$