CALCULUS 2007: EXAM SOLUTIONS

1. (a) The integration region is the triangle in the xy-plane enclosed by the lines $x = 1$, $y = 0$ and $y = 3x$. $\boxed{3}$

Changing the order of integration we obtain $\boxed{3}$

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=3x} x e^{x^3}\, dy.$$

The integral in y gives $\boxed{2}$

$$\int_{y=0}^{y=3x} x e^{x^3}\, dy = \left[y x e^{x^3} \right]_0^{3x} = 3x^2 e^{x^3}.$$

Plugging this result into the second integral we obtain $\boxed{2}$

$$I = \int_{x=0}^{x=1} 3x^2 e^{x^3}\, dx = \left[e^{x^3} \right]_0^1 = e - 1.$$

The last integral can be easily carried out using the change of variables $t = x^3$.

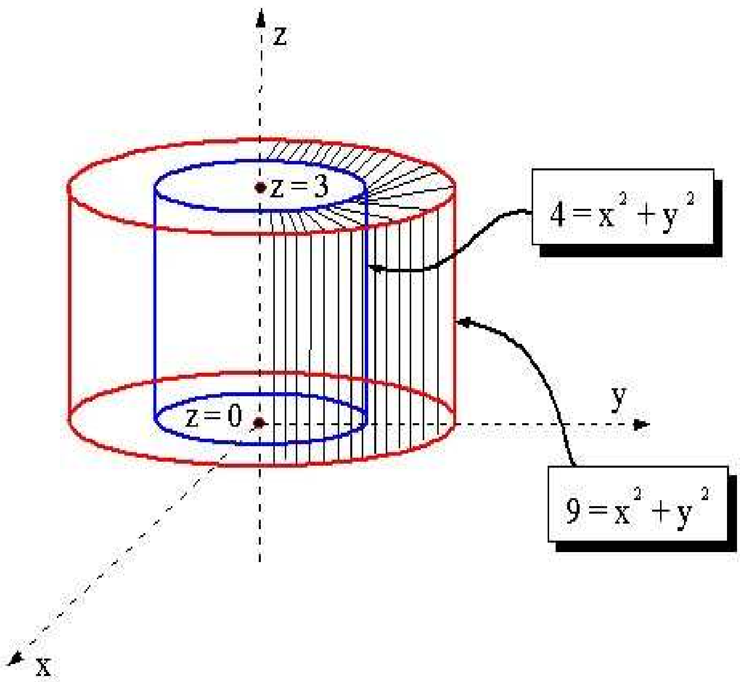
(b) The Jacobian of the change of coordinates is simply $\boxed{2}$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, the element of volume which we need to use for the integral is $\boxed{1}$

$$dx\, dy\, dz = |J|\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz.$$

The integration region for this problem is quite easy to sketch. We have two circular cylinders of radii 2 and 3 centered at the origin extending between $z = 0$ and $z = 3$. The volume we want to compute is half of the volume contained between the two cylinders and the given planes. It is half, because the problem says that $y > 0$. The integration region is therefore the dashed volume in the picture below



In cylindrical coordinates, the integration region is simply 4

$$R = \{(r, z, \theta) : 2 \leq r \leq 3, \quad 0 \leq z \leq 3, \quad 0 \leq \theta \leq \pi\},$$

and the integral we want to compute is therefore 1

$$V = \int_{r=2}^{r=3} r dr \int_{\theta=0}^{\theta=\pi} d\theta \int_{z=0}^{z=3} dz.$$

The various integrals can be carried out separately and give 1

$$\int_{r=2}^{r=3} r dr = \left[\frac{r^2}{2}\right]_2^3 = \frac{5}{2}, \quad \int_{\theta=0}^{\theta=\pi} d\theta = \pi, \quad \int_{z=0}^{z=3} dz = 3.$$

Therefore 1

$$V = (\pi)(5/2)(3) = \frac{15\pi}{2}.$$

2. (a) First of all we need to find the points at which the first order partial derivatives vanish. These derivatives are 1

$$f_x = 2ax + by, \qquad f_y = 2cy + bx.$$

Then 1

$$f_x = 0 \quad \Leftrightarrow \quad x = -\frac{by}{2a},$$

and $\boxed{1}$

$$f_y = 0 \quad \Leftrightarrow \quad x = -\frac{2cy}{b}.$$

The two derivatives vanish simultaneously only if $x = y = 0$. Therefore, we have a single point to study: $(0, 0)$. To investigate what type of stationary point this point is, we have to look at the second order partial derivatives: $\boxed{2}$

$$f_{xx} = 2a, \qquad f_{yy} = 2c,$$

$$f_{xy} = b.$$

For the point $(0, 0)$ we find that:

$$f_{xx}f_{yy} - f_{xy}^2 = 4ac - b^2 > 0,$$

and since $f_{xx} > 0$ we conclude that the point is a minimum. $\boxed{3}$

If $a < 0$, then

$$f_{xx}f_{yy} - f_{xy}^2 = 4ac - b^2 < 0,$$

and the point would be a saddle point. $\boxed{2}$

(b) The Taylor expansion of a function of two variables $f(x, y)$ around a point (x_0, y_0) up to second order terms is given by $\boxed{2}$

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0, y_0) = (0, 0)$ and $\boxed{2}$

$$f_x = e^{x+y}(1+x)\,y, \quad f_y = e^{x+y}x\,(1+y)\,, \quad f_{xx} = e^{x+y}(2+x)\,y,$$

$$f_{yy} = e^{x+y}x\,(2+y)\,, \quad f_{xy} = f_{yx} = e^{x+y}(1+x)(1+y)\,.$$

Therefore $\boxed{2}$

$$f(0,0) = 0, \quad f_x(0,0) = 0, \quad f_y(0,0) = 0, \quad f_{xx}(0,0) = 0,$$

$$f_{yy}(0,0) = 0, \quad f_{xy}(0,0) = f_{yx}(0,0) = 1.$$

Hence the Taylor expansion is just $\boxed{2}$

$$f(x, y) = xy.$$

Since $f_x(0, 0) = f_y(0, 0) = 0$ we know that the point is an stationary point of the function. In addition we have that

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = -1 < 0,$$

and therefore $(0, 0)$ is a saddle point of $f(x, y)$. $\boxed{2}$

3. (a) Since $G = 0$ also its total differential $dG = 0$ must vanish. By definition

$$dG = G_x dx + G_y dy + G_z dz = 0,$$

and in addition, z is a function of x and y, therefore its differential is given by

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy.$$

If we substitute dz into dG we obtain the equation

$$dG = 0 = \left(G_x + G_z \frac{\partial z}{\partial x}\right) dx + \left(G_y + G_z \frac{\partial z}{\partial y}\right) dy.$$

Since x and y are independent variables, the equation above implies that each of the factors has to vanish separately, that is

$$G_x + G_z \frac{\partial z}{\partial x} = G_y + G_z \frac{\partial z}{\partial y} = 0.$$

Therefore we obtain,

$$\frac{\partial z}{\partial x} = -\frac{G_x}{G_z}, \qquad \frac{\partial z}{\partial y} = -\frac{G_y}{G_z}.$$

[6]

Employing now these formulae for $G(x, y, z) = \sin(xyz)$ we obtain

$$\frac{\partial z}{\partial x} = -\frac{z}{x}, \qquad \frac{\partial z}{\partial y} = -\frac{z}{y}.$$

[4]

(b) First, we compute the partial derivatives of f and ϕ, where $\phi(x, y) = x^2 - y^2 - 1 = 0$: [1]

$$f_x = 6, \quad f_y = 1, \quad \phi_x = 2x, \quad \phi_y = -2y.$$

Therefore, the system of equations which we need to solve is: [2]

$$x^2 - y^2 - 1 = 0,$$

$$6 + 2\lambda x = 0,$$

$$1 - 2\lambda y = 0.$$

From the two last equations, we obtain: [2]

$$\lambda = \frac{1}{2y}, \qquad \lambda = -\frac{3}{x},$$

therefore

$$\frac{1}{2y} = -\frac{3}{x} \quad \Leftrightarrow \quad x = -6y.$$

Plugging this constraint into $\phi(x, y) = 0$ we obtain: [2]

$$y^2(36-1) = 1 \quad \Leftrightarrow \quad y = \pm\frac{1}{\sqrt{35}}.$$

The corresponding values of λ are $\boxed{1}$

$$\lambda = \pm\frac{\sqrt{35}}{2}.$$

The maximum value of $f(x,y)$ corresponds to $y = -1/\sqrt{35}$ and $x = 6/\sqrt{35}$, with $\lambda = \sqrt{35}/2$ and is $\boxed{2}$

$$f(6/\sqrt{35}, -1/\sqrt{35}) = \sqrt{35},$$

and the minimum value is

$$f(-6/\sqrt{35}, 1/\sqrt{35}) = -\sqrt{35}.$$

4. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition $\boxed{2}$

$$m^2 + 8m + 25 = 0 \Rightarrow m = -4 \pm 3i.$$

This means that the general solution of the homogeneous equation is of the form $\boxed{2}$

$$y = e^{-4x}(c_1 \sin(3x) + c_2 \cos(3x)),$$

therefore we identify

$$u_1(x) = e^{-4x}\sin(3x), \qquad u_2(x) = e^{-4x}\cos(3x).$$

For the second part of the problem we will need the Wronskian of these solutions which is $\boxed{3}$

$$\begin{aligned} W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-4x}\sin(3x) & e^{-4x}\cos(3x) \\ -4e^{-4x}\sin(3x) + 3e^{-4x}\cos(3x) & -4e^{-4x}\cos(3x) - 3e^{-4x}\sin(3x) \end{vmatrix} \\ &= -4e^{-8x}\sin(3x)\cos(3x) - 3e^{-8x}\sin^2(3x) + 4e^{-8x}\sin(3x)\cos(3x) - 3e^{-8x}\cos^2(3x) = -3e^{-8x}. \end{aligned}$$

Therefore the Wronskian is indeed nowhere zero (for finite values of x).

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form $\boxed{3}$

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = e^{-2x}, \qquad W(x) = -3e^{-8x},$$

therefore $\boxed{4}$

$$v_1(x) = \frac{1}{3}\int e^{2x}\cos(3x)dx.$$

Integrating by parts twice, it is possible to prove that

$$v_1(x) = \frac{e^{2x}\left(2\cos(3x) + 3\sin(3x)\right)}{39}.$$

and 4

$$v_2(x) = -\frac{1}{3}\int e^{2x}\sin(3x)dx = \frac{-e^{2x}\left(-3\cos(3x) + 2\sin(3x)\right)}{39},$$

which can be obtained also integrating by parts. Hence the general solution of the inhomogeneous equation is 2

$$\begin{aligned} y &= e^{-4x}(c_1\sin(3x) + c_2\cos(3x)) + \sin(3x)\frac{e^{-2x}\left(2\cos(3x) + 3\sin(3x)\right)}{39} \\ &- \cos(3x)\frac{e^{-2x}\left(-3\cos(3x) + 2\sin(3x)\right)}{39} = e^{-4x}(c_1\sin(3x) + c_2\cos(3x)) + \frac{e^{-2x}}{13}. \end{aligned}$$

with c_1, c_2 being arbitrary constants.