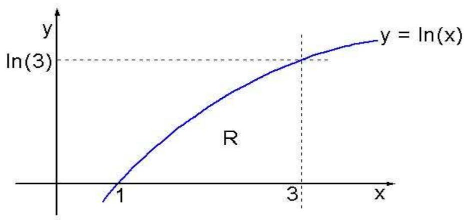
CALCULUS 2010: EXAM SOLUTIONS

1. (a) *[Seen with same integration region but different integrand]* The integration region is:



3 points

Reversing the order of integration we obtain

$$I = \int_{y=0}^{y=\ln(3)} dy \int_{x=e^y}^{x=3} (x+y)dx.$$

The x-integral gives 3 points

$$\int_{x=e^y}^{x=3} (x+y)dx = \left[\frac{x^2}{2} + yx\right]_{x=e^y}^{x=3} = \frac{9-e^{2y}}{2} + 3y - ye^y.$$

Carrying out the final integral we obtain, 2 points

$$I = \frac{1}{2}\int_{y=0}^{y=\ln(3)} (9 - e^{2y} + 6y - 2ye^y)dy = \frac{1}{2}\left[9y - \frac{e^{2y}}{2} + 3y^2\right]_{y=0}^{y=\ln(3)} - \underbrace{\int_{y=0}^{y=\ln(3)} ye^y dy}_{\text{by parts once}}$$

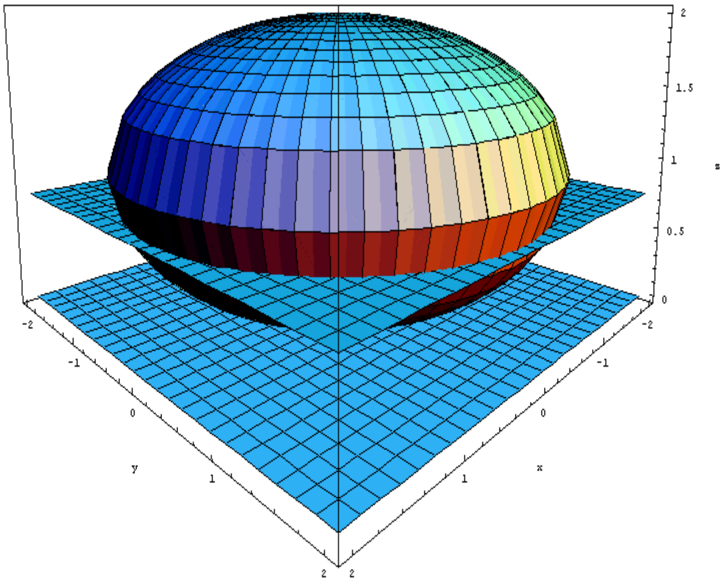
$$= \frac{1}{2}\left(9\ln(3) - \frac{e^{2\ln(3)}}{2} + \frac{1}{2} + 3(\ln(3))^2\right) - \underbrace{\int_{y=0}^{y=\ln(3)} ye^y dy}_{\text{by parts once}} = \frac{1}{2}(9\ln(3) - 4 + 3(\ln(3))^2) - \underbrace{\int_{y=0}^{y=\ln(3)} ye^y dy}_{\text{by parts once}}$$

$$= \frac{1}{2}(9\ln(3) - 4 + 3(\ln(3))^2) - [ye^y]_{y=0}^{y=\ln(3)} + \int_{y=0}^{y=\ln(3)} e^y dy$$

$$= \frac{1}{2}(9\ln(3) - 4 + 3(\ln(3))^2) - 3\ln(3) + 2 = \frac{3\ln(3)}{2}(1 + \ln(3)).$$

4 points

(b) *[A very similar exercise was done as coursework]* The picture of the three surfaces involved is given below:



From the picture (*not required in the exam*) and the information given by the problem, it is clear that the variable z in the integration region takes values $0 \leq z \leq 3/4$. Also, it is easy to see that $0 \leq \theta \leq 2\pi$, since the ellipsoid is centred at the z axis (at $z = 1$). The only variable for which the integration limits are not obvious is r. The values r can take must be determined by the ellipsoid's equation, which in cylindrical coordinates takes the form:

$$4r^2 + (z-1)^2 = 1 \quad \Rightarrow \quad r = \frac{1}{2}\sqrt{1-(z-1)^2}.$$

Therefore $0 \leq r \leq \frac{1}{2}\sqrt{1-(z-1)^2}$. **[2.5 points]**

The integral that we have to compute is:

$$V = \int_{\theta=0}^{\theta=2\pi} d\theta \int_{z=0}^{z=3/4} dz \int_{r=0}^{1/2\sqrt{1-(z-1)^2}} r dr,$$

where we included the Jacobian $J = r$ for cylindrical coordinates. The integral in r is: **[2.5 points]**

$$\int_{r=0}^{1/2\sqrt{1-(z-1)^2}} r dr = [r^2/2]_{r=0}^{r=1/2\sqrt{1-(z-1)^2}} = \frac{1-(z-1)^2}{8}.$$

Plugging this into the z-integral we obtain **[1 points]**

$$\frac{1}{8}\int_{z=0}^{z=3/4} (1-(z-1)^2) dz = \frac{1}{8}[z - \frac{(z-1)^3}{3}]_{z=0}^{z=3/4} = \frac{1}{8}\left(\frac{3}{4} + \frac{1}{192} - \frac{1}{3}\right) = \frac{27}{512}.$$

Finally, the integral in θ is: **[2 points]**

$$\int_{\theta=0}^{\theta=2\pi} d\theta = [\theta]_{\theta=0}^{\theta=2\pi} = 2\pi.$$

Therefore

$$V = \pi \frac{27}{256} \approx 0.33134.$$

1 points

1 points

2. (a)[*Not seen*] First of all we need to find the points at which the first order partial derivatives vanish. These derivatives are 1 point

$$f_x = (1 + (x - y)y)e^{xy}, \qquad f_y = (-1 + (x - y)x)e^{xy}.$$

Then

$$f_x = 0 \quad \Leftrightarrow \quad 1 + (x - y)y = 0,$$

and

$$f_y = 0 \quad \Leftrightarrow \quad -1 + (x - y)x = 0.$$

Subtracting the two equations from each other we find

$$2 + (x - y)(y - x) = 0 \quad \Rightarrow \quad 2 - (x - y)^2 = 0 \quad \Rightarrow \quad x - y = \pm\sqrt{2}.$$

2 points

Substituting $x = y \pm \sqrt{2}$ in equation $f_x = 0$ we obtain

$$1 \pm \sqrt{2}y = 0 \quad \Rightarrow \quad y = \mp 1/\sqrt{2}.$$

Therefore we get two stationary points:

$$(x, y) = (-\frac{1}{\sqrt{2}} + \sqrt{2}, -\frac{1}{\sqrt{2}}) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}),$$

and

$$(x, y) = (\frac{1}{\sqrt{2}} - \sqrt{2}, \frac{1}{\sqrt{2}}) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

2 points

To investigate what type of stationary points this points are, we have to look at the second order partial derivatives: 2 points

$$f_{xx} = (2y + (x - y)\,y^2)e^{xy}, \qquad f_{yy} = (-2x + (x - y)\,x^2)e^{xy},$$

$$f_{xy} = f_{yx} = (x - y)(2 + xy)e^{xy}.$$

Calling $A = f_{xx}$, $B = f_{xy}$ and $C = f_{yy}$, we find:

i) For the point $(1/\sqrt{2}, -1/\sqrt{2})$ we have

$$A = -\frac{1}{\sqrt{2e}}, \qquad B = \frac{3}{\sqrt{2e}}, \qquad C = -\frac{1}{\sqrt{2e}}.$$

Then

$$AC - B^2 = -\frac{4}{e} < 0,$$

therefore this point is a saddle point. 1.5 points

ii) For the point $(-1/\sqrt{2}, 1/\sqrt{2})$ we have

$$A = \frac{1}{\sqrt{2e}}, \qquad B = -\frac{3}{\sqrt{2e}}, \qquad C = \frac{1}{\sqrt{2e}}.$$

Then

$$AC - B^2 = -\frac{4}{e} < 0,$$

therefore this point is also a saddle point. 1.5 points

(b) [*A very similar problem was done as coursework*] The function that we want to maximize and minimize is the distance between a point (x, y) on the sphere and the point $(0, 0)$. The distance function is

$$d(x, y) = \sqrt{x^2 + y^2}.$$

Since the derivatives of this function are rather complicated, it is better to consider the square of the distance instead. This is correct because both the distance and its square will be maximal or minimal at the same points! So, let us consider the function

$$f(x, y) = x^2 + y^2.$$

The constrain in this case is given by the fact that (x, y) must lie on the circle:

$$\Phi(x, y) = (x - 2)^2 + (y + 1)^2 - 4 = 0.$$

1 point

According to the method of Lagrange multipliers, we need to solve the following set of equations:

$$\begin{aligned} f_x + \lambda\Phi_x &= 0 \quad \Rightarrow \quad 2x + \lambda 2(x - 2) = 0 && (0.1) \\ f_y + \lambda\Phi_y &= 0 \quad \Rightarrow \quad 2y + \lambda 2(y + 1) = 0 && (0.2) \\ (x - 2)^2 + (y + 1)^2 - 4 &= 0 && (0.3) \end{aligned}$$

2 points

The two first equations can be re-written as:

$$\frac{x}{x-2} = \frac{y}{y+1} = -\lambda,$$

by solving each equation for λ. This implies that:

$$x(y + 1) = y(x - 2) \quad \Rightarrow \quad xy + x = yx - 2y \quad \Rightarrow \quad x = -2y.$$

1 point

Substituting this result, into equation (0.3) we find an equation for y which we can solve:

$$(-2y-2)^2+(y+1)^2-4 = 0 \quad \Rightarrow \quad 5(y+1)^2-4 = 0 \quad \Rightarrow \quad y+1 = \pm\sqrt{\frac{4}{5}} \quad \Rightarrow \quad y = -1\pm\frac{2}{\sqrt{5}}.$$

1.5 points

We therefore have two solutions for y which correspond to two solutions for x (since $x = -2y$) given by $x = 2 \mp \frac{4}{\sqrt{5}}$. Therefore we find two points on the sphere which are at maximal or minimal distance from $(0, 0)$ and they are:

$$(x, y) = \left(2 - \frac{4}{\sqrt{5}}, -1 + \frac{2}{\sqrt{5}}\right) \quad \text{and} \quad \left(2 + \frac{4}{\sqrt{5}}, -1 - \frac{2}{\sqrt{5}}\right).$$

1.5 points

We still need to find the corresponding values of the Lagrange multiplier. For this we can just substitute the two different values of x for each point onto equation (0.1). This gives:

$$-\lambda = \frac{x}{x-2} = \frac{2 \mp \frac{4}{\sqrt{5}}}{\mp \frac{4}{\sqrt{5}}} = \mp \frac{\sqrt{5}}{2} + 1,$$

1 point

where the minus sign corresponds to the first point and the plus sign to the second point.

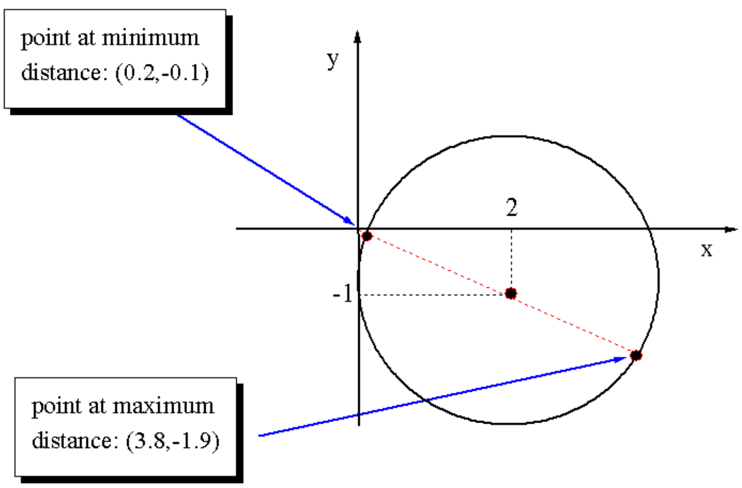
Now the only thing left to do is to find out which one of these points is a t maximum distance and which one at minimum distance. For this we just need to substitute the two solutions into the function $d(x, y)$ that we found above. We find:

$$d\left(2 - \frac{4}{\sqrt{5}}, -1 + \frac{2}{\sqrt{5}}\right) = \sqrt{\left(2 - \frac{4}{\sqrt{5}}\right)^2 + \left(-1 + \frac{2}{\sqrt{5}}\right)^2} = \sqrt{9 - 4\sqrt{5}} = 0.236068...$$

and

$$d\left(2 + \frac{4}{\sqrt{5}}, -1 - \frac{2}{\sqrt{5}}\right) = \sqrt{\left(2 + \frac{4}{\sqrt{5}}\right)^2 + \left(-1 - \frac{2}{\sqrt{5}}\right)^2} = \sqrt{9 + 4\sqrt{5}} = 4.23607...$$

Clearly the distance is largest for the second point and therefore this provides the maximum whereas the first point is the one at minimum distance. The figure below shows the location of the points. 2 points



From this picture we actually see that the two points are connected by a line that includes the centre of the sphere and the origin of coordinates and that the largest distance must be equal to the shortest distance plus the diameter of the sphere (4), which is precisely what we got!

3. (a) [*Not seen*] Using the chain rule we have

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = (v+3u^2)f_x + vf_y,$$

and

2 points

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = uf_x + (u+3v^2)f_y.$$

The 2nd order partial derivatives can be obtained by using once more the chain rule: 2 points

$$\begin{aligned}\frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}((v+3u^2)f_x + vf_y) = 6uf_x + (v+3u^2)f_{xu} + vf_{yu}\\ &= 6uf_x + (v+3u^2)((v+3u^2)f_{xx} + vf_{xy}) + v((v+3u^2)f_{xy} + vf_{yy})\\ &= 6uf_x + (v+3u^2)^2 f_{xx} + 2v(v+3u^2)f_{xy} + v^2 f_{yy},\end{aligned}$$

and

3 points

$$\begin{aligned}\frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v}(uf_x + (u+3v^2)f_y) = uf_{xv} + 6vf_y + (u+3v^2)f_{yv}\\ &= u(uf_{xx} + (u+3v^2)f_{xy}) + 6vf_y + (u+3v^2)(uf_{xy} + (u+3v^2)f_{yy})\\ &= u^2 f_{xx} + 2u(u+3v^2)f_{xy} + 6vf_y + (u+3v^2)^2 f_{yy}.\end{aligned}$$

3 points

(b) [*Not seen*] We saw in the lecture that for an implicit function defined by an equation of the form:

$$\Phi(x,y,z) = 3yz^2 - e^{4xz} - 3y^2 + 4 = 0,$$

1 point

The derivatives may be computed as:

$$\frac{\partial z}{\partial x} = -\frac{\Phi_x}{\Phi_z} = \frac{4ze^{4zx}}{6yz - 4xe^{4zx}} = \frac{2ze^{4zx}}{3yz - 2xe^{4zx}},$$

2 points

$$\frac{\partial z}{\partial x} = -\frac{\Phi_y}{\Phi_z} = -\frac{3z^2 - 6ye^{4zx}}{6yz - 4xe^{4zx}}.$$

2 points

At the point $(1,0)$ we have

$$\Phi(1,0,z) = -e^{4z} + 4 = 0, \qquad z = \frac{1}{4}\ln 4 = \ln\sqrt{2} \approx 0.346574$$

2 points

Therefore the value of the derivatives is:

$$z_x(1,0) = \frac{8\ln\sqrt{2}}{-8} = -\ln\sqrt{2} \approx -0.346574$$

1.5 points

$$z_y(1,0) = -\frac{3(\ln\sqrt{2})^2}{-8} = \frac{3}{8}(\ln\sqrt{2})^2 \approx 0.0450425$$

1.5 points

4. *[Not seen]* To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 - 9 = 0 \Rightarrow m = \pm 3.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 e^{3x} + c_2 e^{-3x},$$

therefore we identify

$$u_1(x) = e^{3x}, \qquad u_2(x) = e^{-3x}.$$

2 points

For the second part of the problem we will need the Wronskian of these solutions which is

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 3e^{2x} & -3e^{-2x} \end{vmatrix} = -3 - 3 = -6.$$

2 points

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

3 points

In our case

$$R(x) = \frac{6}{\sinh(3x)}, \qquad W(x) = -6,$$

therefore, integrating by parts twice we obtain

$$v_1(x) = \int \frac{e^{-3x}}{\sinh(3x)}dx = 2\int \frac{e^{-3x}}{e^{3x} - e^{-3x}}dx = 2\int \frac{e^{-9x}}{1 - e^{-9x}}dx = \frac{2}{9}\ln|1 - e^{-9x}|$$

5 points

$$v_2(x) = -\int \frac{e^{3x}}{\sinh(3x)}dx = -2\int \frac{e^{3x}}{e^{3x} - e^{-3x}}dx = -2\int \frac{e^{9x}}{e^{9x} - 1}dx = -\frac{2}{9}\ln|e^{9x} - 1|.$$

Hence the general solution of the inhomogeneous equation is [5 points]

$$y = c_1 e^{3x} + c_2 e^{-3x} + \frac{2e^{3x}}{9} \ln |1 - e^{-9x}| - \frac{2e^{-3x}}{9} \ln |e^{9x} - 1|.$$

with c_1, c_2 being arbitrary constants. [3 points]