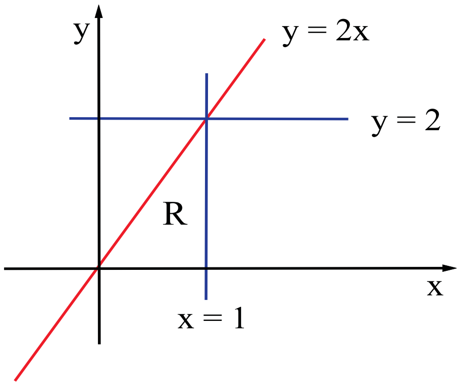
CALCULUS 2006: EXAM SOLUTIONS

1. (a) The integration region is the lower triangle in the picture 3



From the picture it is easy to see that changing the order of integration we obtain 3

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=2x} e^{x^2}\, dy.$$

The integral in y gives 2

$$\int_{y=0}^{y=2x} e^{x^2}\, dy = \left[y e^{x^2} \right]_0^{2x} = 2x e^{x^2}.$$

Plugging this result into the second integral we obtain 2

$$I = \int_{x=0}^{x=1} 2x e^{x^2}\, dx = \left[e^{x^2} \right]_0^1 = e - 1.$$

(b) The Jacobian of the change of coordinates is simply 2

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

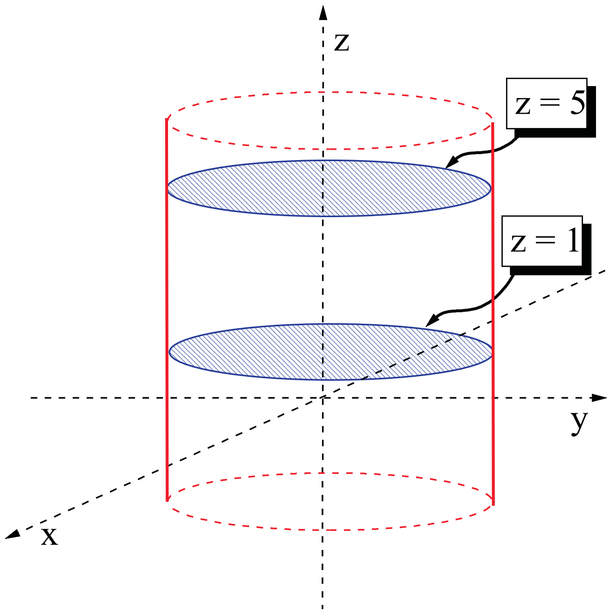
Therefore, the element of volume which we need to use for the integral is 1

$$dx\, dy\, dz = |J|\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz.$$

To compute the integral we have first to express the integrand in terms of the new variables, that is 1

$$(x^2 + y^2)^3 = (r^2)^3 = r^6.$$

The next step is to describe the region of integration in terms of the new variables. The integration region for this problem is very easy to sketch. We have a radius 1 circular cylinder centered at the origin extending between the $z = 1$ and $z = 5$ planes. This looks more or less like in the picture below



In cylindrical coordinates, the integration region is simply 2

$$R = \{(r, z, \theta) : 0 \leq r \leq 1, \quad 1 \leq z \leq 5, \quad 0 \leq \theta \leq 2\pi\},$$

and the integral we want to compute is therefore 1

$$V = \int_{r=0}^{r=1} r^7 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{z=1}^{z=5} dz.$$

The various integrals can be carried out separately and give 2

$$\int_{r=0}^{r=1} r^7 dr \quad = \quad \left[\frac{r^8}{8}\right]_0^1 = \frac{1}{8}, \quad \int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi, \quad \int_{z=1}^{z=5} dz = 5 - 1 = 4.$$

Therefore 1

$$V = (2\pi)(1/8)(4) = \pi.$$

2. (a) First of all we need to find the points at which the first order partial derivatives vanish. These derivatives are 1

$$f_x = 4y - 4x^3 = 0, \qquad f_y = 4x - 4y^3 = 0$$

Then 1

$$f_x = 0 \quad \Leftrightarrow \quad y = x^3,$$

and [1]

$$f_y = 0 \quad \Leftrightarrow \quad x = y^3.$$

Inserting the first condition into the second one we obtain

$$x = x^9 \Rightarrow x^8 = 1 \quad \text{or} \quad x = 0 \Rightarrow \quad x = 1, 0, -1,$$

which gives us 3 candidates to be stationary points, that is the points $(1,1), (0,0)$ and $(-1,-1)$ at which both f_x and f_y vanish. To investigate what type of stationary points this points are, we have to look at the second order partial derivatives: [1]

$$f_{xx} = -12x^2, \qquad f_{yy} = -12y^2,$$

$$f_{xy} = f_{yx} = 4.$$

Calling $A = f_{xx}$, $B = f_{xy}$ and $C = f_{yy}$, we find:

i) For the point $(1,1)$ we have [2]

$$\begin{aligned} AC - B^2 &= 144 - 16 = 128 > 0, \\ A &= -12 < 0, \end{aligned}$$

therefore this point is a **maximum**.

ii) For the point $(0,0)$ we have [2]

$$AC - B^2 = -16 < 0,$$

therefore this point is a **saddle point**.

iii) For the point $(-1,-1)$ we have [2]

$$\begin{aligned} AC - B^2 &= 144 - 16 = 128 > 0, \\ A &= -12 < 0, \end{aligned}$$

therefore this point is also a **maximum**.

(b) The Taylor expansion of a function of two variables $f(x,y)$ around a point (x_0, y_0) up to second order terms is given by [2]

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0, y_0) = (0,1)$ and [2]

$$\begin{aligned} f_x &= (1+x+y)e^{x-y}, \quad f_y = (1-x-y)e^{x-y}, \quad f_{xx} = (2+x+y)e^{x-y}, \\ f_{yy} &= (-2+x+y)e^{x-y}, \qquad f_{xy} = f_{yx} = -(x+y)e^{x-y}. \end{aligned}$$

Therefore [2]

$$\begin{aligned} f(0,1) &= 1/e, \quad f_x(0,1) = 2/e, \quad f_y(0,1) = 0, \quad f_{xx}(0,1) = 3/e, \\ f_{yy}(0,1) &= -1/e, \qquad f_{xy}(0,1) = f_{yx}(0,1) = -1/e, \end{aligned}$$

So, the Taylor expansion is [2]

$$f(x,y) = \frac{1}{2e}(2 + 4x + 3x^2 - (y-1)^2 - 2x(y-1)),$$

and [2]

$$f(0.1, 1.1) = \frac{1}{2e}(2 + 0.4 + 3(0.1)^2 - (0.1)^2 - 2(0.1)^2) = \frac{2.4}{2e}.$$

The exact value of the function at this point is

$$f(0.1, 1.1) = \frac{1.2}{e}.$$

therefore the Taylor approximation is in fact exact at this point!

3. (a) Using the chain rule we have [4]

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = uf_x + vf_y,$$

and [4]

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -vf_x + uf_y.$$

(b) From (a) we can obtain the 2nd order partial derivatives by using once more the chain rule we have [5]

$$\begin{aligned} \frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}(uf_x + vf_y) = f_x + u\frac{\partial f_x}{\partial u} + v\frac{\partial f_y}{\partial u} \\ &= f_x + u(uf_{xx} + vf_{yx}) + v(uf_{xy} + vf_{yy}) = f_x + u^2 f_{xx} + v^2 f_{yy} + 2uvf_{xy}, \end{aligned}$$

and [5]

$$\begin{aligned} \frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v}(-vf_x + uf_y) = -f_x - v\frac{\partial f_x}{\partial v} + u\frac{\partial f_y}{\partial v} \\ &= -f_x - v(-vf_{xx} + uf_{yx}) + u(-vf_{xy} + uf_{yy}) = -f_x + v^2 f_{xx} + u^2 f_{yy} - 2uvf_{xy}. \end{aligned}$$

Substracting the two formulae we trivially see that [2]

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = (u^2 + v^2)(f_{xx} + f_{yy}).$$

4. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition [2]

$$m^2 - 4m + 5 = 0 \Rightarrow m = 2 \pm i.$$

This means that the general solution of the homogeneous equation is of the form 2

$$y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x,$$

therefore we identify

$$u_1(x) = e^{2x} \cos x, \qquad u_2(x) = e^{2x} \sin x.$$

For the second part of the problem we will need the Wronskian of these solutions which is 3

$$\begin{aligned} W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} \cos x & e^{2x} \sin x \\ 2e^{2x} \cos x - e^{2x} \sin x & 2e^{2x} \sin x + e^{2x} \cos x \end{vmatrix} \\ &= e^{4x} \cos x (2 \sin x + \cos x) - e^{4x} \sin x (2 \cos x - \sin x) = e^{4x}. \end{aligned}$$

Therefore the Wronskian is indeed nowhere zero.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form 3

$$y = v_1(x) u_1(x) + v_2(x) u_2(x),$$

with

$$v_1(x) = -\int u_2(x) \frac{R(x)}{W(x)} dx \qquad \text{and} \qquad v_2(x) = \int u_1(x) \frac{R(x)}{W(x)} dx.$$

In our case

$$R(x) = \frac{e^{2x}}{\sin x}, \qquad W(x) = e^{4x},$$

therefore 4

$$v_1(x) = -\int dx = -x,$$

4

$$v_2(x) = \int \frac{\cos x}{\sin x} dx = \ln(\sin x).$$

Hence the general solution of the inhomogeneous equation is 2

$$y = e^{2x} (c_1 \cos x + c_2 \sin x - x \cos x + \ln(\sin x) \sin x),$$

with c_1, c_2 being arbitrary constants.