Solutions to sheet 1

1.
$$f(1,4) = (3)(1)(2) - 1 = 5, \qquad f(0,9) = (3)(0)(3) - 1 = -1,$$
$$f(t^2,t) = (3)(t^4)\sqrt{t} - 1, \qquad f(ab, 9b) = (3)(ab)^2(3\sqrt{b}) - 1.$$
The domain of the function is: $\mathcal{D}(f) = \{(x,y)\,|\,y \geq 0\}$

2.
$$f(0,1/2,-1/2) = \sqrt{1 - 0^2 - (1/2)^2 - (-1/2)^2} = \sqrt{1-1/2} = \sqrt{1/2}.$$
The domain of the function is $\mathcal{D}(f) = \left\{(x,y,z) \quad | \quad x^2+y^2+z^2 \leq 1\right\}$. In this case, since the function depends on 3 variables, the domain lives in $\mathbb{R}^3$ and is the interior and boundary of a ball of radius 1.

3. (a) Is the upper hemisphere of a sphere of radius 1. (b) Is a cone with $z \leq 0$.

4. $\mathcal{D}(f) = \left\{(x,y) \quad | \quad x^2 > y\right\}$. To sketch the domain you just have to draw the parabola $y = x^2$. All points that are outside the region enclosed by the parabola will have $x^2 > y$.

5. (a) This set contains all points inside a disk of radius 1 in the xy-plane excluding the origin $(0,0)$:
$$\text{Boundary} = \left\{(x,y)\,|\,x^2+y^2 = 1\right\}, \qquad \text{Interior} = \left\{(x,y)\,|\,0 < x^2+y^2 < 1\right\}.$$
$$\text{Exterior} = \left\{(x,y)\,|\,x^2+y^2 > 1 \quad \text{and} \quad (x,y) = (0,0)\right\}.$$
The set does not contain its boundary and therefore it is <u>open</u>.

(b) This set is just a "quarter" of the $\mathbb{R}^2$ space corresponding to $x \geq 0$ and $y < 0$, including the line $x = 0$ with $y < 0$.
$$\text{Boundary} = \{(x,y)\,|\,x = 0 \quad \text{with} \quad y \leq 0 \quad \text{or} \quad y = 0 \quad \text{with} \quad x \geq 0\}.$$
$$\text{Interior} = \{(x,y)\,|\,x > 0 \quad \text{and} \quad y < 0\}, \qquad \text{Exterior} = \{(x,y)\,|\,x < 0 \quad \text{or} \quad y > 0\}.$$
The set contains part of the boundary but not all the points on the boundary, therefore it is <u>neither open nor closed</u>.

(c) This set is just the line $y = 1 - x$.
$$\text{Boundary} = \{(x,y)\,|\,x + y = 1\}, \qquad \text{Interior} = \{\emptyset\}, \qquad \text{Exterior} = \{(x,y)\,|\,x + y \neq 1\}.$$
In this case the set not only contains the boundary but coincides with it. Therefore it is <u>closed</u>.

(d) In this case the set is a square in $\mathbb{R}^2$ formed by the intersection of the lines $y = 1 - x$, $y = -1 - x$, $y = 1 + x$ and $y = -1 + x$.
$$\text{Boundary} = \{(x,y)\,|\,|x| + |y| = 1\}, \qquad \text{Interior} = \{\emptyset\}, \qquad \text{Exterior} = \{(x,y)\,|\,|x| + |y| \neq 1\}.$$
As in the previous case, the set coincides with its boundary and therefore it is <u>closed</u>.

6. **(a)** the limit exists and is 4, **(b,e,h)** the limit does not exist, **(c,d,f)** the limit exists and is 0, **(g)** the limit exists and is 1/4.

7. **Example 1:** All functions here are continuous.

Examples 2 and 3: Is not continuous at $(0,0)$ since the limit does not exist.

Example 4: Is not continuous at $(0,0)$ since the value of the function at $(0,0)$ is not defined, although the limit exists at that point. The function can be made continuous by re-defining it as:
$$f_4(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

8. The limit exists and is 0 if $m + n > 2p$. This can be proven by using a similar argument as in example 4 of page 12 (done in the lecture).